Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0

CLASSIFICATION CONFIDENTIAL CONFIDENTIAL

50X1-HUM

CENTRAL INTELLIGENCE AGENCY
INFORMATION FROM
FOREIGN DOCUMENTS OR RADIO BROADCASTS

REPORT

CD NO.

COUNTRY	USSR	DATE OF INFORMATION	1951
SUBJECT	Scientific - Mathematics, Bessel functions	DATE DIST.	17 Sep 1951
HOW PUBLISHED	Thrice-monthly periodical		
WHERE PUBLISHED	Moscow	NO. OF PAGES	5
DATE PUBLISHED	1 Apr 1951	SUPPLEMENT TO REPORT NO.	
LANGUAGE	Russian		

THIS DOCUMENT CONTAINS INFORMATION AFFECTING THE NATIONAL DEFENSE OF THE UNITED STATES WITHIN THE MEANING OF ESPIONAGE ACT 50 U. S. C. 31 AND 32, AS AMENDED. ITS TRANSMISSION OR THE REVELATION OF ITS CONTENTS IN ANY MANNER TO AN UNAUTHORIZED PERSON IS PROHIBITED BY LAW. REPRODUCTION OF THIS FORM IS PROHIBITED.

THIS IS UNEVALUATED INFORMATION

SOURCE Doklady Akademii Nauk SSSR, Vol LXXVII, No 4, 1951, pp 561-564.

COMPUTATION OF SUMS OF NEGATIVE EVEN POWERS OF ROOTS OF BESSEL FUNCTIONS

Acad P. L. Kapitsa

In the computation of values of roots of Bessel functions and in the solutions of certain problems of mathematical physics it is necessary to know how to compute negative even powers of these roots.

Euler gave a direct method for the computation of these sums, which consists in expanding the Bessel function in terms of powers of the variable and in equating them to the same function, represented in the form of an infinite product, containing the values of roots. In this identity the coefficients of the same powers of the variable are equal, and this enables us to compute the unknown sums. In practice, this method leads to complicated computations. To facilitate this computation, Rayleigh [1] took logarithms of both parts of the identity, using the expansion $\ln(1+x)$ and presenting it in the form of a series. Thus he was able to compute the sum up to the 10th power. We present here a less direct but considerable simpler method of computing these sums. Besides, the same method may be applied to computation of the root sum of some ordinary equations containing Bessel functions.

Let $\lambda_1, \lambda_2 \ldots$ be the successive values of roots of the Bessel function; we shall designate the unknown sums by:

$$\sigma_\nu^{(r)} = \sum_{k=1}^{k=\infty} \lambda_k^{-2r}; \quad J_\nu(\lambda_k) = 0. \tag{1}$$

Let us expand the Bessel function $J_{\nu-1}(\beta x)$ into Dini [?] series in functions $J_{\nu-1}(\lambda_k x)$. The series will have the following form:

$$J_{\nu-1}(\beta x) = a_0 x^{\nu-1} + \sum_{k=1}^{k=\infty} a_k J_{\nu-1}(\lambda_k x). \tag{2}$$

CLASSIFICATION CONFIDENTIAL CONFIDENTIAL

STATE	X	NAVY	X	NSRB		DISTRIBUTION						
ARMY	X	AIR	X	FBI								

Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0

Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0

50X1-HUM

In the present problem the role of the term with a_0 is rather essential, because, as is known [2, p 655], this term is indispensable in the expansion. The coefficient a_0 is determined by:

$$a_0 = 2\nu \int_0^1 J_{\nu-1}(\beta x)\, x^{\nu} dx = 2\nu \beta^{-1} J_{\nu}(\beta); \tag{3}$$

the coefficient a_k will be:

$$a_k = 2\int_0^1 J_{\nu-1}(\beta x) J_{\nu-1}(\lambda_k x)\, x dx \Big/ \int_0^1 J^2_{\nu-1}(\lambda_k x)\, x dx. \tag{4}$$

The recurrent formula for $J_{\nu-1}(\lambda_k)$ taking (1) into account will be:

$$J_{\nu-1}(\lambda_k) = \frac{\nu-1}{\lambda_k} J_{\nu-1}(\lambda_k) \tag{5}$$

Taking this expression into account and applying the usual recurrent ratios, we finally get

$$J_{\nu-1}(\beta x) = 2\nu\beta^{-1} J_{\nu}(\beta) x^{\nu-1} - 2\beta J_{\nu}(\beta) \sum_{k=1}^{k=\infty} \frac{1}{\lambda_k^2 - \beta^2} \frac{J_{\nu-1}(\lambda_k x)}{J_{\nu-1}(\lambda_k)}. \tag{6}$$

Assuming in this expression $x=1$ and $\beta \to 0$, we obtain, after simple reduction, the familiar expression [1]

$$\sigma_{\nu}^{(1)} = \frac{1}{2^2(\nu+1)} \tag{7}$$

To obtain sums of higher order, we multiply the terms of expansion (6) by $x^{\nu}dx$, we integrate from 0 to x, thereafter multiply again by xdx, integrate from 0 to x, and so on. after having integrated m+1 times, we assume x=1. We obtain

$$\beta^{-\nu-1} J_{\nu+m}(\beta) = \frac{\beta^{-1} J_{\nu}(\beta)}{2^m(\nu+1)\cdots(\nu+m)} - 2\beta J_{\nu}(\beta) \sum_{k=1}^{k=\infty} \frac{1}{\lambda_k^{m+1}(\lambda_k^2 - \beta^2)} \frac{J_{\nu+m}(\lambda_k)}{J_{\nu-1}(\lambda_k)}. \tag{8}$$

As far as λ_k is a root of the Bessel function of ν-th order, we can write, by using Lommel's [2, p 322] polynomials,

$$J_{\nu+m}(\lambda_k) = -J_{\nu-1}(\lambda_k) R_{m-1,\nu+1}(\lambda_k). \tag{9}$$

By substituting this value in the expansion, we obtain after some reduction, assuming $\beta \to 0$

$$\frac{\sigma_m \Gamma(\nu+1)}{2^{2m+2}(\nu+1)\Gamma(\nu+m+2)} =$$

$$= \sum_{k=1}^{k=\infty} \frac{1}{\lambda_k^{2m+2}} \sum_{n=0}^{n \leqslant \frac{1}{2}(m-1)} (-1)^n \frac{(m-1-n)!\,\Gamma(\nu+m-n)}{2^{2n} n!\,(m-1-2n)!\,\Gamma(\nu+n+1)} \lambda_k^{2n}. \tag{10}$$

This expression is valid for $m \geqslant 1$, and assuming m taking the values 1, 2, 3, ... successively, we obtain $\sigma_{\nu}(r)$ for all even powers, starting with four.

Thus, we get all values obtained by Rayleigh. Besides, we still can compute the following values, not computed by the previous method:

Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0

Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0

50X1-HUM

$$\sigma_\nu^{(6)}=\frac{21\nu^3+181\nu^2+513\nu+473}{2^{11}(\nu+1)^6(\nu+2)^3(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)},$$

$$\sigma_\nu^{(7)}=\frac{33\nu^3+329\nu^2+1081\nu+1145}{2^{12}(\nu+1)^7(\nu+2)^3(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)(\nu+7)}. \qquad (11)$$

The sum of $\sigma_\nu^{(8)}$ was computed by Keli [Kelley?] [2, p 554] by a method which may be applied only in the case if r is a power of two. We repeated Keli's results by our method and in addition computed $\sigma_\nu^{(9)}$ and $\sigma_\nu^{(10)}$ which we present

$$\sigma_\nu^{(9)}=\frac{715\nu^6+16567\nu^5+158568\nu^4+798074\nu^3+2217079\nu^2+3212847\nu+1893046}{2^{17}(\nu+1)^9(\nu+2)^4(\nu+3)^3(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)(\nu+9)}, \qquad (12)$$

$$\sigma_\nu^{(10)}=\frac{2431\nu^8+80425\nu^7+1152851\nu^6+9315667\nu^5+46240675\nu^4+143917279\nu^3+273583653\nu^2+289891557\nu+130934438}{2^{19}(\nu+1)^{10}(\nu+2)^5(\nu+3)^3(\nu+4)^2(\nu+5)^3(\nu+6)(\nu+7)(\nu+8)(\nu+9)(\nu+10)}.$$

These values can be checked by assuming $\nu=1/2$; then, these sums are expressed in Bernoulli's numbers [2, p 554]. For large values of the exponential r, the magnitude of the sum is determined by the first root λ_1 with sufficient accuracy for a number of problems, and this root may be simply computed by extracting the root of 2r power from the whole sum.

It is known [2] that the expansion of Dini is possible in roots δ_k of the following equation:

$$\delta_k J_\nu'(\delta_k)-HJ_\nu(\delta_k)=0, \quad \nu+1/2\geqslant 0. \qquad (13)$$

Let us designate

$$\eta_\nu^{(r)}=\sum_{k=1}^{k=\infty}\delta_k^{-2r} \qquad (14)$$

These sums may be found by the same method by expanding in a series. By expanding in roots of equation (13) it is necessary to distinguish three cases; in the second and third cases it is necessary to introduce different terms with a_0. These cases are the following:

$$a)\ \nu>H\geqslant 0; \qquad b)\ \nu=H; \qquad c)\ \nu<H \qquad (15)$$

We shall discuss the first case when $\nu>H\geqslant 0$; it is the simplest case because in the expansion (2) the term with a_0 is absent (3). Let us expand the function $J_\nu(\beta x)$ in $J_\nu(\delta_k x)$, after usual reductions the series will take the aspect:

$$J_\nu(\beta x)=2[\beta J_{\nu-1}(\beta)-(\nu+H)J_\nu(\beta)]\sum_{k=1}^{k=\infty}\frac{\delta_k^2}{(\delta_k^2-\beta^2)[\delta_k^2-(\nu^2-H^2)]}\frac{J_\nu(\delta_k x)}{J_\nu(\delta_k)}. \qquad (16)$$

Let us assume x=1 and $\beta\to 0$, then after reduction we obtain:

$$\varphi_\nu^{(0)}=\sum_{k=1}^{k=\infty}\frac{1}{\delta_k^2-(\nu^2-H^2)}=\frac{1}{2(\nu-H)}. \qquad (17)$$

We multiply the expansion (16) by $x^{\nu+1}dx$ and integrate from 0 to x, then multiply again by xdx and again integrate from 0 to x, and so on m times. We obtain by assuming x = 1:

$$\beta^{-m}J_{\nu+m}(\beta)=$$

$$=2[\beta J_{\nu-1}(\beta)-(\nu+H)J_\nu(\beta)]\sum_{k=1}^{k=\infty}\frac{1}{\delta_k^{m-2}(\delta_k^2-\beta^2)[\delta_k^2-(\nu^2-H^2)]}\frac{J_{\nu+m}(\delta_k)}{J_\nu(\delta_k)} \qquad (18)$$

Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0

Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0

50X1-HUM

From expression (13) we obtain by using the recurrent ratio

$$J_{\nu-1}(\delta_k)=\frac{\nu+H}{\delta_k}J_\nu(\delta_k). \tag{19}$$

Using this equality, we obtain the following relation between Bessel functions of various orders, expressed in Lommel's polynomials:

$$J_{\nu+m}(\delta_k)=[R_{m,\nu}(\delta_k)-\frac{\nu+H}{\delta_k}R_{m-1,\nu+1}(\delta_k)]J_\nu(\delta_k). \tag{20}$$

Substituting this value in (18) and assuming $\beta\to 0$, we obtain after transformation

$$\frac{\Gamma(\nu+1)}{\nu^{m}(\nu-H)\Gamma(m+\nu+1)}=\sum_{k=1}^{k=\infty}\frac{1}{\delta_k^{2m}[\delta_k^2-(\nu^2-H^2)]}\sum_{n=0}^{n<m/2}(-1)^n[m(\nu-H)+ \tag{21}$$
$$+2n(H+m-n)]\frac{(m-n-1)!\,\Gamma(\nu+m-n)}{n!(m-2n)!\,\Gamma(\nu+n+1)}\left(\frac{\delta_k}{2}\right)^{2n}.$$

From this expression by successive computations we obtain a series of expressions of the following type

$$\varphi_\nu^{(r)}=\sum_{k=1}^{k=\infty}\frac{1}{\delta_k^{2r}[\delta_k^2-(\nu^2-H^2)]}. \tag{22}$$

The unknown sums are found the following way.

$$\eta_\nu^{(r)}=\varphi_\nu^{(r-1)}-(\nu^2-H^2)\,\varphi_\nu^{(r)}. \tag{23}$$

I present several sums $\eta_\nu^{(r)}$, computed by this method:

$$\eta_\nu^{(1)}=\frac{\nu+2-H}{2^2(\nu+1)(\nu-H)};\qquad \eta_\nu^{(2)}=\frac{4(\nu+1)+(\nu+2-H)^2}{2^4(\nu+1)^2(\nu+2)(\nu-H)^2}; \tag{24}$$
$$\eta_\nu^{(3)}=\frac{4(4+3\nu-H)(\nu+1)(\nu+2)(\nu+3)-(\nu+H)(4+3\nu-H)^2(\nu+3)-(\nu+H)(\nu-H)^2(\nu+1)}{2^6(\nu+1)^3(\nu+2)^2(\nu+3)(\nu-H)^3}.$$

Of practical interest is the case when $H=0$, then, the roots δ_k become roots of the derivative of the Bessel function. In this case the expressions $\eta_\nu^{(r)}$ are considerably simplified.

In the presented second case of (15) for the conditions for expansion when $\nu=H$, the roots δ_k become roots of the Bessel function of $\nu+1$ order, which corresponds to the problem initially discussed.

In the third case of (15), when $\nu<H$, equation (13) possesses an additional two imaginary roots $\pm i\delta_0$ [2, p 531]. It follows from the expansion theory [2] that in this case the term a_0 also equals zero and is expressed by the magnitude δ_0. Following the previous examples in a similar way, we may here too obtain the sum $\eta_\nu^{(r)}$.

It is of interest to remark that the sums $\sigma_\nu^{(r)}$ and $\eta_\nu^{(r)}$ may also be obtained by expanding x^ν into a series in functions $J_{\nu-1}(\lambda_k x)$ and $J_{\nu-1}(\delta_k x)$. This is the method by which Rayleigh [3], in one of his problems with spherical resonators, obtained the value $\varphi_{1/2}^{(0)}$ according to our designation, and Lamb [4] generalized this result for $\varphi_\nu^{(0)}$ (17). These works had no further development although the computation of roots of this kind of transcendental equations was within the range of interests of mathematical physics of that time. Probably the lag in the development of this method was due to the fact that at that time the value of the additional term with a_0 in the expansion into a Dini series had not yet been found. Much later, in 1908, the need for introducing these terms was pointed out by Bridgeman [5]; however, it is not to be found in later manuals. To obtain the limited result of Lamb or Rayleigh, the absence of the additional term had no effect.

Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0

Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0

CONFIDENTIAL

CONFIDENTIAL

50X1-HUM

BIBLIOGRAPHY

1. J. W. S. Rayleigh, Sci Papers, 1, 190 (1899)
2. G. N. Watson, Theory of Bessel Functions, Party I, 1949
3. J. W. S. Rayleigh, The Theory of Sound, 1896, 332
4. H. Lamb, Proc London Math Soc, 15, 273 (1884)
5. P. W. Bridgeman, Phil Mag, 16, 947 (1908)

- E N D -

CONFIDENTIAL

CONFIDENTIAL

Sanitized Copy Approved for Release 2011/09/21 : CIA-RDP80-00809A000700010251-0