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"APPROXIMATE SOLUTION OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS"

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The DE: $\ddot{x} + h(x,\dot{x})\dot{x} + k^2(x,\dot{x})x = 0$ cannot be integrated in known functions and must be solved by the approximation method of 'averaging'. The most widespread variant of this method is Vander Pol's method, in which one sets $x = \alpha \sin(\omega t + \varepsilon)$ where omega w is a certain constant frequency and α, ε are slowly varying functions of time t, connected by the relation: $\dot{\alpha} \sin(\omega t + \varepsilon) + \alpha\dot{\varepsilon} \cos(\omega t + \varepsilon) = 0$ for $\dot{\alpha}$ and $\dot{\varepsilon}$ one obtains the "shortened" equations averaged over the period $2\pi/w$:

$$\dot{\alpha} = -\frac{\alpha}{2\pi}\int_0^{2\pi} h(\alpha \sin u, \alpha\omega \cos u) \cos^2 u \, du$$

$$\dot{\varepsilon} = \frac{1}{2\pi\omega}\int_0^{2\pi} k^2(\alpha \sin u, \alpha\omega \cos u) \sin^2 u \, du - \frac{\omega}{2}$$

Here ε and α are found by quadrature.

Let us take the partial case where h and k^2 depend only on x. then $\dot{\alpha}$ becomes:

$$\dot{\alpha} = -\frac{\alpha}{2\pi}\int_0^{2\pi} h(\alpha \sin u) \cos^2 u \, du$$

and gives the instantaneous amplitude of α as a funtion of time.

Similarly for $\dot{\varepsilon}$:

$$\omega + \dot{\varepsilon} = \frac{\omega}{2} + \frac{1}{2\pi\omega}\int_0^{2\pi} k^2(\alpha \sin u) \sin^2 u \, du$$

which determines the instantaneous frequency $w + \dot{e}$.

If one can choose the constant w so that, in the course of the entire process, $\dot{\varepsilon}$ has remained a small quantity relative to w, then $w + \dot{\varepsilon}$ can be expressed thus:

$$\omega^2 + 2\omega\dot{\varepsilon} = \frac{1}{\pi}\int_0^{2\pi} k^2(\alpha \sin u) \sin^2 u \, du$$

or, approximately:

$$(\omega + \dot{\varepsilon})^2 \doteq \frac{1}{\pi}\int_0^{2\pi} k^2(\alpha \sin u) \sin^2 u \, du \quad (\text{since } \dot{\varepsilon}^2 \text{ is small})$$

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namely an expression for the square of the instantaneous frequency $\omega + \dot{\varepsilon}$, whose right side does not depend upon w.

If the frequency of the process varies considerably with time, then $\dot{\varepsilon}$ becomes of the same order as w and the above operation is vitiated (addition of $\dot{\varepsilon}^2$ on the left side only).

Utilization of the original equation for $\omega + \dot{\varepsilon}$ leads to a physically incoherent result; namely, the instantaneous frequency $\omega + \dot{\varepsilon}$ turns out to be dependent upon the choice of the constant ω and is always greater than $\frac{1}{2} \cdot w$.

Such results are explained by the fact that if $\dot{\varepsilon}$ is arranged to be small then ε cannot be considered a slowly varying function of time [sic!]; consequently one of the basic assumptions is violated and becomes useless. This is also true even for the more accurate variant of the Van der Pol method of averaging given by Bulgakov (Oscillations, GTTI, 1949, Vol I).

Let $x = \alpha \sin\left(\int_0^t \omega\, dt + \varepsilon\right)$, where $\alpha, \omega, \varepsilon$ are slowly varying functions of time; here $\dot{\omega}$ also will be considered a slowly varying function of time. Let us impose the following condition:

$$\dot{\alpha} \sin\left(\int_0^t \omega\, dt + \varepsilon\right) + \alpha \dot{\varepsilon} \cos\left(\int_0^t \omega\, dt + \varepsilon\right) = 0$$

and set $x = \alpha \sin\left(\int_0^t \omega\, dt + \varepsilon\right)$ into the original DE. To obtain:

$$\alpha\left\{k^2\left[\alpha \sin\tau(t), \alpha\omega\cos\tau(t)\right] - \omega^2 - \omega\dot{\varepsilon}\right\}\sin\tau(t) +$$
$$+ \left\{\alpha\dot{\omega} + \dot{\alpha}\omega + \alpha\omega h\left[\alpha\sin\tau(t), \alpha\omega\cos\tau(t)\right]\right\}\cos\tau(t) = 0 \; ; \; \tau(t) \equiv \int_0^t \omega\, dt + \varepsilon .$$

Determining $\dot{\alpha}$ and $\dot{\varepsilon}$ from these two equations and averaging the obtained formulas over the period of the trigonometric functions entering them, we find:

$$\dot{\alpha} = -\frac{\alpha\dot{\omega}}{2\pi} - \frac{\alpha}{2\pi}\int_0^{2\pi} h(\alpha\sin u, \alpha\omega\cos u)\cos^2 u\, du$$

$$\dot{\varepsilon} = \frac{1}{2\pi\omega}\int_0^{2\pi} k^2(\alpha\sin u, \alpha\omega\cos u)\cdot\sin^2 u\, du - \frac{\omega}{2} .$$

We require now that ω be averaged over the period of the circular frequency of the process and, consequently, that in the average over the period $\dot{\varepsilon} = 0$.

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Then for the equation just above for $\dot{\varepsilon}$ we obtain

$$\omega^2 = \frac{1}{\pi}\int_0^{2\pi} k^2(\alpha \sin u, \alpha\omega \cos u)\sin^2 u\,du,$$

and the equation for $\dot{\alpha}$ assumes after the substitution $\dot{\omega} = \frac{d\omega}{d\alpha}\dot{\alpha}$ the following form:

$$\dot{\alpha}\left(1+\frac{\alpha}{2\omega}\cdot\frac{d\omega}{d\alpha}\right) = -\frac{\alpha}{2\pi}\int_0^{2\pi} h(\alpha \sin u, \alpha\omega \cos u)\cos^2 u\,du.$$

Next, ω is determined as a function of α and then substituted in the last equation, which then becomes a DE in α and t and integrable by quadrature.

[A simple exam le of the DE for a regulation system with a nonlinear servomotor is given and solved easily by the new method just described.]

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