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THE LINEAR BOUNDARY PROBLEM FOR THE UNSTEADY MOTION OF A VISCOUS INCOMPRESSIBLE FLUID

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After the works of Odqvist (1, 2), Lichtenstein (3), and Leray (4), dedicated to proving the theorem of the existence and uniqueness of a solution of the boundary problem for the steady motion of a viscous liquid with small Reynolds number, this problem can be considered completely developed. Two works of Odqvist (5) and Leray (6) may be cited which treat boundary problems for unsteady motion.

A solution of the linear problem is given in Odqvist's work with some rather general assumptions by two different methods depending upon whether a finite or infinite field is being examined. The problem for a plane is similarly examined in Leray's extensive work. Leray reduces the solution of the linear problem on a plane to a system of two singular integral equations. Using the method of successive approximation, Leray also examines the nonlinear problem on a plane and shows that a single solution of the so-called partially linearized problem does exist.

The present work is dedicated to the solution of the basic linear boundary problem for the unsteady motion of a viscous incompressible liquid. The problem consists of the following: determination of regular intrafield solutions of the linear hydrodynamic equations with given values of velocity at the boundary at the initial moment. The field

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is considered to be a single compendency, unaffected by time, and limited by an arbitrary closed surface with a continuously variable tangential plane and primary curvature. If the field is external, the desired solutions must also satisfy the given conditions for infinity. The solutions are considered constant inside the field and have continuous derivatives up to a certain order dependent upon the equations. The given initial values are also regular inside the field, and unless specifically stipulated otherwise, we shall demand only continuity from the boundary values.

A solution of the linear problem in the internal as well as the external field is given here. In contrast to Odqvist (6), after proof of the theorem on the uniqueness of the solution and reduction of the initial conditions to zero, we set up fundamental solutions by which the problem is reduced to the solution of a system of quasi-regular integral equations of the second kind. The solution of this system is found by successive approximation, simultaneously showing its uniquenessness. The solution of the plane problem arising out of the general linear case is given at the end of the paper.

I. FORMULATION OF THE PROBLEM AND UNIQUENESS OF THE SOLUTION

1. Let a viscous incompressible liquid found in an unsteady state of motion fill an internal or external field D, limited by a solid closed regular surface F. Let x_1, x_2, and x_3 be the coordinates of a point in the field; t the time; v (v_1, v_2, v_3) the vector of velocity; p the hydrodynamic pressure; ρ the density of the liquid; and ν the kinematic coefficient of viscosity.

In the absence of external forces, the linear boundary problem is formulated in the following form: determination of the functions v and p, the regular intrafield motions which satisfy the linear hydrodynamic equations

$$\nu \Delta v - \frac{\partial v}{\partial t} = \frac{1}{\rho}\, \mathrm{grad}\, p, \quad \mathrm{div}\, v = 0 \qquad \left(\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\right) \tag{1.1}$$

with the limiting conditions: when t > 0 the velocity v is given on the boundary, and at the initial moment v is a given function of the coordinates inside the field.

In the case of an external field, without limiting the generality, it can be considered that the motion dies out at infinity, so that if $v_1 = v - v_\infty$ is placed in (1.1) the free member may be included in the expression p, and we will have equations (1.1) and the given boundary conditions with a zero condition at infinity for the desired functions.

2. We shall show that the boundary problem examined here cannot have more than one solution in which the hydrodynamic pressure is determined with accuracy up to an arbitrary function depending upon the time. For this it is sufficient to show that the system (1.1) with zero limiting conditions has only a zero solution.

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Multiplying both parts of the first equation of system (1.1) scalarly by v and taking the integral in the field D, we have

$$\frac{d}{dt}\int_D v^2 dD = 2\nu\int_D v\cdot\Delta v\, dD - \frac{1}{\rho}\int_D v\cdot \operatorname{grad} p\, dD \tag{1.2}$$

By the second equation of system (1.1) we have

$$v\cdot \operatorname{grad} p = \operatorname{div}(pv) - p \operatorname{div} v = \operatorname{div}(pv) \tag{1.3}$$

and according to the Gauss formula, we obtain

$$\int_D v\cdot \operatorname{grad} p\, dD = -\int_F p v_n\, dF = 0 \tag{1.4}$$

where n is the internal normal to the surface F, and v_n is equal to zero on the surface. Formula (1.4) is also applicable in the case of an external field since the velocity at infinity equals zero, and we will consider that $\lim Rp v = 0$ with $R \to \infty$.

Also, according to Green's formula, it is possible to write (considering $\partial v/\partial n$ to be continuous in the field D + F)

$$\int_D \sum (\operatorname{grad} v_i)^2 dD = -\int_F v\cdot\frac{\partial v}{\partial n} dF - \int_D v\cdot\Delta v\, dD$$

Since v becomes zero on the surface, we obtain

$$\int_D v\cdot\Delta v\, dD \leq 0 \tag{1.5}$$

On the basis of the formulas (1.4) and (1.5), equation (1.3) gives

$$\frac{d}{dt}\int_D v^2 dD \leq 0$$

which proves our hypothesis. Acutally, at the initial moment v, as well as the integral along the field D from v^2, are equal to zero; therefore, the examined integral cannot decrease in the consequent because it is not negative. On the other hand, if this integral is

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not equal to zero, then because of the latter formula of its derivative according to time, it must be negative throughout, which is impossible. Consequently, the vector v is exactly equal to zero.

3. The examined boundary problem can be reduced to a boundary problem with initial conditions of zero. We shall present v as the sum of two solutions v_0 and v_*, and have the vector v_0 satisfy (1.1) and the initial conditions.

To find the vector v_0 we shall represent it as the vortex of a new vector V; i.e., assume $v_0 = \text{rot } V$, and substitute it in system (1.1); after eliminating the pressure we obtain for vector V the equation

$$\nu \Delta\Delta V - \frac{\partial \Delta V}{\partial t} + \frac{\partial}{\partial t} \text{grad div } V - \nu \Delta \text{ grad div } V = 0 \tag{1.6}$$

Since vector V is determined with accuracy up to a gradient of an arbitrary function, we may take div V 0 without destroying the generality. Actually, in the expression of the vector V an arbitrary vector grad W may be included. Then to satisfy the latter condition we will have

$$V = V_1 + \text{grad } W, \qquad \Delta W = -\text{div } V_1 \tag{1.7}$$

Thus, equation (1.6) can be presented in the form

$$\nu \Delta\Delta V - \frac{\partial \Delta V}{\partial t} = 0$$

The vortex of vector V is given at the initial moment, but obviously

$$(\text{rot } V)_{t=0} = \text{rot } (V)_{t=0}$$

From this condition the vector $V_0 = (V)_{t=0}$ may be determined. Actually, in conformity with the condition div V = 0, we assume that with regard to the limiting value of V, $V_n = 0$ on the surface F. Then the problem of the determination of V_0 will consist of the following: determination of the vector of velocity V_0 of an incompressible liquid in a field bounded by a closed stationary surface F, when the vortex rot V_0 is given inside the field and $V_n = 0$ on the boundary; or, if the field is not limited, $V_0 = 0$ at infinity. This problem is solved by the usual methods of classical hydrodynamics (7). Therefore the initial value V can be considered as given.

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It is easy to prove directly that the partial solution of equation (1.7) may be presented in the form

$$V' = \frac{1}{8(\sqrt{\pi\nu t})^3}\int_{-\infty}^{+\infty} d\xi_1 \int_{-\infty}^{+\infty} d\xi_2 \int_{-\infty}^{+\infty} V(\xi_1,\xi_2,\xi_3,0)\exp\left[\frac{-1}{4\nu t}\sum (x_i-\xi_i)^2\right] d\xi_3$$

We shall show that vector V' satisfies the initial condition. Assuming $\xi_i = x_i + 2\alpha_i\sqrt{\nu t}$, we will have

$$V' = \frac{1}{\sqrt{\pi^3}}\int_{-\infty}^{+\infty} d\alpha_1 \int_{-\infty}^{+\infty} d\alpha_2 \int_{-\infty}^{+\infty} V(x_1+2\alpha_1\sqrt{\nu t},\dots,0)\exp\left[-\sum\alpha_i^2\right] d\alpha_3$$

Shifting to the limit with $t \longrightarrow 0$, we obtain

$$(V')_{t=0} = V(x_1,x_2,x_3,0)$$

Therefore, a solution v_0 of system (1.1) will be found, which satisfies the given initial conditions and the condition of attenuation at infinity (in the external field) because $v_0 =$ rot V' satisfies (1.1) as well as the indicated conditions. Therefore it remains to find a solution v_* of system (1.1) which satisfies the given limiting conditions, with the initial value equal to zero.

Consequently, the boundary problem under examination may be formulated as follows: determination of the function v_i (i = 1, 2, 3), and of p, which are regular in the field examined and satisfy the equations

$$\nu\Delta v_i - \frac{\partial v_i}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial x_i}, \qquad \sum \frac{\partial v_i}{\partial x_i} = 0 \tag{1.8}$$

and the limiting conditions

$$(v_i)_F = f_i \quad (r > 0); \quad (v_i)_{t=0} = 0 \tag{1.9}$$

and in the case of the external field, also satisfy the condition of attenuation at infinity. Also, we shall consider it established that, according to conditions (1.8) and (1.9), the function v_i will be found as simple numbers, and p accurate to an arbitrary function related to the time (if they exist).

II. FUNDAMENTAL SOLUTIONS

1. First, we shall reproduce briefly Odqvist's results for the solution of the steady linear problem (1) according to his work mentioned above (2).

Let $P(x_1, x_2, x_3)$ be an internal point in the field; u_i the components of the velocity u, and q the hydrodynamic pressure of the steady motion. The functions u_i and q, which are regular inside the field, satisfy the linear system of equations

$$\nu \Delta u_i = \frac{1}{\rho} \frac{\partial q}{\partial x_i}, \quad \sum \frac{\partial u_i}{\partial x_i} = 0$$

(2.1)

At the boundary u_i takes given continuous values; in the case of an infinite field, u_i satisfies the condition of attenuation at infinity.

We shall designate by $\omega_j(M)$ $(j = 1, 2, 3)$ the arbitrary continuous functions of the point $M(\xi_1, \xi_2, \xi_3)$ of the surface F. We present the solution of (2.1) in the form

$$u_i = \int_F \sum_j \omega_j(M) u_{ij}(P, M)\, dF_M$$

$$q = -\int_F \sum_j \omega_j(M) q_j(P, M)\, dF_M \qquad (2.2)$$

where

$$u_{ij} = \frac{3\cos(r,n)}{2\pi r^4}(x_i - \xi_i)(x_j - \xi_j)$$

$$q_i = \frac{\nu\rho}{\pi} \frac{\partial}{\partial x_j} \frac{\cos(r,n)}{r^2}$$

$$r^2 = \sum (x_i - \xi_i)^2$$

in which vector r runs from point M to point P.

The expressions u_i, determined by formulae (2.2), are characterised by properties analogous to the properties of the potential of a double layer. They are regular inside the field, but on crossing through the boundary surface, they experience a discontinuity of the first order, according to which a system of integral equations is obtained for determination of the unknown functions ω_j,

$$\omega_i(N) + \lambda \int_F \sum \omega_j(M) u_{ij}(N, M)\, dF_M = u_i(N) \qquad (2.3)$$

where N is the limiting position of the point P with respect to its streamline to the boundary surface, $u_i(N)$ are the given boundary values of the components of velocity, and $\lambda = +1$ in the case of an internal field, or $\lambda = -1$ if the field is external.

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System (2.3) presents a quasi-regular system of three Fredholm integral equations. Using the usual theory, it is established: with $\lambda = -1$ system (2.3) has a unique continuous solution for continuous $u_i(N)$; with $\lambda = +1$ it is necessary and sufficient for the existence of a solution that the given boundary values of velocity through the surface F be equal to zero.

If the motion is two-dimensional, the indices i and j in the formulas presented will change from 1 to 2, and for u_{ij} and q_j we shall take the values

$$u_{ij} = \frac{\cos(r,n)}{r^3}(x_i - \xi_i)(x_j - \xi_j), \qquad q_j = \frac{\partial}{\partial x_j}\frac{\cos(r,n)}{r}$$

Then the solution of the problem is presented in the form

$$u_i = \frac{2}{\pi}\int_C \sum_j \omega_j(M) u_{ij}(P,M)\, ds$$
$$q = -\frac{2\nu}{\rho}\int_C \sum_j \omega_j(M) q_j(P,M)\, ds \tag{2.4}$$

where C is the contour bounding the examined plane field, and ds is the element of the arc of this contour at point M.

To determine the functions ω_j we will have a system of two integral equations

$$\omega_i(N) + \lambda\int_C \sum_j \omega_j(M) u_{ij}(N,M)\, ds = u_i(N) \tag{2.5}$$

A solution of the system (2.5) in the internal field ($\lambda = +1$), as well as in the three-dimensional case, exists if $u_i(N)$ satisfies the condition that the transfer of velocity through the contour C be equal to zero.

In the external field ($\lambda = -1$), in contrast to the three-dimensional case, a solution of (2.5) will exist only for those boundary values $u_i(N)$ which are orthogonal to the solutions of the allied homogeneous system. Fulfillment of this condition, generally speaking, is impossible, and it is necessary to assume that u_i becomes logarithmically infinite at infinity.

2. To develop the fundamental solutions of the unsteady motion problem, we will use the fundamental solutions of the steady motion problem. We introduce the expressions

$$\Phi_i = -\frac{x_i - \xi_i}{2\pi r^3}\sum n_j(x_j - \xi_j), \quad U_{ik} = \frac{n_k(x_i - \xi_i)}{2\pi r^3} + \frac{\partial \Phi_i}{\partial x_k} \tag{2,6}$$

where n_i is the directing cosine of the normal n with the coordinate axis x_i.

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Then the functions u_{ik} may be determined in the form

$$u_{ik} = \delta_{ik} \frac{\cos \gamma_i}{2\pi r^2} + U_{ik}, \quad \delta_{ik} = \begin{cases} 1 & \text{where } i = k \\ 0 & \text{where } i \neq k \end{cases}$$

where γ is the angle between $r = MP$ and the normal n. It is easy to prove directly that Φ_i and U_{ik} satisfy the equations

$$\Delta \Phi_i = -2 \frac{\partial}{\partial x_i} \frac{\cos \gamma}{2\pi r^2}, \quad \sum_{k=1}^{3} \frac{\partial U_{ik}}{\partial x_k} = -\frac{\partial}{\partial x_i} \frac{\cos \gamma}{2\pi r^2}$$

We shall examine a combination of the following nine values:

$$v_{ik} = \delta_{ik} A(P, M, t) + V_{ik}(P, M, t)$$

(2.7)

where

$$A = \frac{r \cos \gamma}{8\sqrt{(\pi\nu)^3 t^5}} \exp \frac{-r^2}{4\nu t}, \quad V_{ik} = \frac{n_k (x_i - \xi_i)}{8\sqrt{(\pi\nu)^3 t^5}} \exp \frac{-r^2}{4\nu t} + \frac{\partial \varphi_i}{\partial x_k}$$

(2.8)

with φ_i satisfying Poisson's equation and the boundary condition

$$\Delta \varphi_i = -2 \frac{\partial A}{\partial x_i}, \quad \varphi_i(N, M, t) = \frac{r^3 \Phi_i(N, M)}{4\sqrt{\pi \nu^3 t^5}} \exp \frac{-r^2}{4\nu t}.$$

(2.9)

Function A satisfies the equation of thermal conductivity

$$\nu \Delta A - \frac{\partial A}{\partial t} = 0$$

(2.10)

and is regular in field D + F with t > 0, and with t = 0 reverts to zero inside the field. The function φ_i is regular in field D, and by (2.9) we may consider that it reverts to zero in the initial moment. [The function $\varphi_i(N,M,t)$ is characteristic when N agrees with M and t = 0; however, as will be seen from the following paragraph, φ_i and its derivatives along the coordinates revert to zero inside the field with t = 0.] With t > 0, in view of the regularity of A due to the second equation of (2.9), the function φ_i also will be regular in field D + F.

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By means of formulas (2.7), (2.8), and (2.9), it is easy to prove directly that v_{ik} satisfy the equation

$$\sum_K \frac{\partial v_{iK}}{\partial x_K} = 0 \tag{2.11}$$

and are regular throughout the closed field $D + F$ with $t > 0$, and at the initial moment revert to zero inside the field. Let us introduce the value

$$p_i = \rho \left(\nu \Delta \varphi_i - \frac{\partial \varphi_i}{\partial t} \right) \tag{2.12}$$

We can show by substitution that v_{ik} and p_i satisfy (1.8). We will call the function v_{ik} and p_i the fundamental solutions of problems (1.8) and (1.9).

III. INTEGRAL EQUATIONS

Introducing the arbitrary continuous functions w_1, w_2, w_3 of the point M and of time, by formulas (2.11), (2.12), and (1.6), we may ~~present the desired components v_i of the velocity and the pressure~~ p in the form

$$v_i(P,t) = \int_F dF \int_0^t \sum w_K(M,\tau) v_{Ki}(P,M,t-\tau) d\tau \tag{3.1}$$

$$p(P,t) = \rho \int_F dF \int_0^t \sum_K w_K(M,\tau) \left(\nu \Delta \varphi_K - \frac{\partial \varphi_K}{\partial \tau} \right) d\tau + p_0(t) \tag{3.2}$$

where $p_0(t)$ is an arbitrary function.

According to equations (2.11) and (2.12), the expressions (3.1) and (3.2) satisfy system (1.8). In addition, they revert to zero at the initial moment (considering $p_0(0) = 0$), and in the case of an external field also satisfy the condition of the attenuation of motion at infinity. It remains to satisfy the boundary conditions (1.9).

We shall study of the behavior of function (3.1) with a streamline of point P to the surface. We have

$$\int_0^t V_{ik} d\tau = \int_0^t \left[V_{ik} - \frac{r^3 U_{ik}}{4\sqrt{\pi\nu^3(t-\tau)^5}} \exp \frac{-r^2}{4\nu(t-\tau)} \right] d\tau +$$

$$+ U_{ik} \int_0^t \frac{r^3}{4\sqrt{\pi\nu^3(t-\tau)^5}} \exp \frac{-r}{4\nu(t-\tau)} d\tau \qquad (3.3)$$

Substituting $t-\tau = r^2/4\nu\omega^2$, we find

$$\left| \int_0^t \frac{r^3}{4\sqrt{\pi\nu^3(t-\tau)^5}} \exp \frac{-r^2}{4\nu(t-\tau)} d\tau \right| < 1$$

Therefore, the second member in the right side of formula (3.3) remains a smaller modulus than $|U_{ik}|$.

On the other hand, according to Odqvist's results, we can maintain that the integral varies continuously through the entire range from $U_{ik}(P, M)$ along the surface F. From this we conclude that the surface integral from the second member in the right side of formula (3.3) will be continuous when crossing through the boundary surface.

Consequently, the question is reduced to the study of the first member in the right side of formula (3.3). We shall indicate it by J_{ik}. Using the second formula of (2.8) and expressions (2.6) for U_{ik} introducing the expression

$$\psi_i(P, M, t) = \frac{r^3 \Phi_i(P, M)}{4\sqrt{\pi\nu^3 t^5}} \exp \frac{-r^2}{4\nu t}$$

the expression for J_{ik} can be presented in the form

$$J_{ik} = \int_0^t \frac{\partial}{\partial x_k}(\varphi_i - \psi_i) d\tau + \frac{(x_k - \xi_i)\Phi_i}{4\sqrt{\pi\nu^3}} \int_0^t \frac{6\nu r(t-\tau) - r^3}{2\nu\sqrt{(t-\tau)^7}} \exp \frac{-r^2}{4\nu(t-\tau)} d\tau$$

The second integral in the right side of the latter formula is indicated by J.

It can easily be shown that substituting $t-\tau = r^2/4\nu\omega^2$, and integrating by parts of the second integral gives

$$J = -\frac{2r}{\sqrt{t^3}} \exp \frac{-r^2}{4\nu t} d\tau$$

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Thus, we have

$$J_{ik} = \int_0^t \frac{\partial}{\partial x_k}(\varphi_i - \psi_i)\,d\tau - \frac{r(x_k - \xi_k)\Phi_i}{2\sqrt{\pi \nu^3 t^3}} \exp\frac{-r^2}{4\nu t} \tag{3.4}$$

According to expression (2.2) for $_i$, the surface integral from the second member in the right side of formula (3.4) will be continuous in a closed field $D + F$ with each $t > 0$. For estimating the first member, we notice that the difference $\varphi_i - \psi_i$ satisfies the equation

$$\Delta(\varphi_i - \psi_i) = B_i, \quad B_i(P, M, t) = \frac{r^3 \Phi_i}{8\sqrt{\pi \nu^5 t^5}}\left(5 - \frac{r^2}{4\nu t}\right)\exp\frac{-r^2}{4\nu t} \tag{3.5}$$

The function B_i reverts to zero at the boundary with $t > 0$, and at the initial moment it is equal to zero inside the field.

Let $Q(y_1, y_2, y_3)$ be an internal point in the field. Designating the Green harmonic function by $G(P, Q)$, we may present the solution of equation (3.5) in the form

$$\varphi_i - \psi_i = \int_D B_i(Q, M, t)\, G(P, Q)\, dD_Q$$

From this we obtain

$$\int_0^t (\varphi_i - \psi_i)\,d\tau = \int_D G(P, Q)\,dD_Q \int_0^t B_i(Q, M, t - \tau)\,d\tau \tag{3.6}$$

Using expression (3.5) for B_i, it is easily shown by integration by parts that

$$\int_0^t B_i\,d\tau = \frac{r^3 \Phi_i}{4\sqrt{\pi \nu t^3}} \exp\frac{-r^2}{4\nu t} \tag{3.7}$$

Expression (3.7) is regular in field $D + F$ with $t > 0$. Therefore, according to formula (3.6), we can conclude that

$$\int_0^t \frac{\partial}{\partial x_k}(\varphi_i - \psi_i)\,d\tau$$

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is a continuous function throughout the range up to the surface. Therefore, J_{ik} is continuous throughout the entire field $D+F$, and by formula (3.3) we arrive at the conclusion that

$$\int_F dF \int_0^{t} V_{ik}\, d\tau$$

also remains continuous with the stream-line of point P toward the surface F.

Consequently,

$$\lim_{P\to N} \int_F dP \int_0^{t} V_{ik}(P, M, t-\tau)\, d\tau = \int_F dF \int_0^{t} V_{ik}(N, M, t-\tau)\, d\tau$$

On the other hand, we have

$$\int_F dF \int_0^{t} w_k V_{ki}\, d\tau = \int_F dF \int_0^{t} [W_k(M,\tau) - W_k(N,t)] V_{ki}\, d\tau + w_k(N,t) \int_F dF \int_0^{t} V_{ki}\, d\tau$$

By virtue of the continuity of w_k, the second as well as the first member in the right side of the latter formula is continuous throughout the entire range. From this it follows that the upper and lower limits of the examined integral, with the stream-line of the point P toward the surface, are equal to its value at a point on the surface, i.e., to the integral

$$\int_F dF \int_0^{t} w_k(M,\tau) V_{ki}(N, M, t-\tau)\, d\tau$$

Consequently, according to formulas (2.7) and (3.1), the behavior of the function v_i near the surface will be characterized by the expression

$$W_i(P, t) = \int_F dF \int_0^{t} w_i(M,\tau) R(P, M, t-\tau)\, d\tau$$

which presents the spatial thermal potential of a double layer with a density w_i. According to the known formulas for the limiting values, by which the upper and lower limits of this potential (see Myunts (8)) are

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$$W_i^+ = w_i(N,t) + W_i^0, \qquad W_i^- = -w_i(N,t) + W_i^0$$

the limiting values v_i take the form

$$v_i^+ = w_i(N,t) + v_i^0 \qquad v_i^- = -w_i(N,t) + v_i^0$$

On the basis of the latter relations, which satisfy the boundary conditions (1.9), we obtain a system of three integral equations for determining the unknown functions w_i

$$\lambda w_i(N,t) + \int_F dF \int_0^t \sum_K w_K(M,\tau) v_{Ki}(N,M,t-\tau)\, d\tau = f_i(N,t) \qquad (3.8)$$

where $f_i(N, t)$ are the given limiting values of the components of velocity $\lambda = +1$ in the case of the internal problem, and $\lambda = -1$ if the problem is external.

IV. INVESTIGATION OF THE SOLUTION

1. Equation (3.8) forms a system of composite integral equations of the Volterra type. The solution of this system can be carried out analogously to the solution of the equation of thermal conductivity by using successive approximations. We shall examine a system with a parameter x

$$\lambda w_i(N,t) + x \int_F dF \int_0^t \sum_K w_K(M,\tau) v_{Ki}(N,M,t-\tau)\, d\tau = f_i(N,t) \qquad (4.1)$$

and present its solution in the form

$$w_i = \sum_{m=0}^{\infty} x^m w_{im} \qquad (4.2)$$

Substituting in system (4.1), we obtain for the determination of the members of the series the recurrent formulas

$$w_{i0} = -\frac{1}{\lambda} f_i;$$

$$w_{i(m+1)} = -\frac{1}{\lambda} \int_F dF \int_0^t \left(w_{im} A + \sum_K w_{Km} V_{Ki} \right) d\tau \qquad (4.3)$$

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In order to study the convergence of series (4.2), we examine the integral

$$J = \int_F dF \int_0^t \left(A + \sum V_{Ki} \right) d\tau \qquad (4.4)$$

Taking into consideration expression (2.8) for A, after permutation of the integrals we will have

$$J = \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \int_F \frac{r \cos\gamma}{8\sqrt{(\pi\nu)^3 (t-\tau)^2}} \exp \frac{-r^2}{4\nu(t-\tau)} dF + \int_F dF \int_0^t \sum V_{Ki}\, dF \qquad (4.5)$$

By formulas (3.3), (3.4) and (3.6), we have

$$\int_F dF \int_0^t V_{Ki}\, d\tau = \int_F \frac{(x_k - \xi_k)(x_i - \xi_i)\cos\gamma}{4r(\sqrt{\pi\nu t})^3} \exp \frac{-r^2}{4\nu t} dF +$$

$$+ \int_F U_{Ki}\, dF \int_0^t \frac{r^3}{4\sqrt{\pi\nu^3 (t-\tau)}^5} \exp \frac{-r^2}{4\nu(t-\tau)} d\tau +$$

$$+ \int_D \frac{\partial G(N,Q)}{\partial x_i} dO \int_F \frac{(y_k - \xi_k)\cos\gamma_{QM}}{8\sqrt{\pi^3 \nu^5 t^5}} \exp \frac{-r^2}{4\nu t} dF \qquad (4.6)$$

Taking into consideration expression (2.6) for U_{Ki}, it is easily noticed that the right side of the latter formula remains bounded by the surface F.

In the vicinity of point N we shall separate an infinitely small part of F_0 of the surface F, and shall designate θ polar coordinates of point M in the plane x_1x_2 by r_0, when the beginning is taken in point N and the axis x_3 is directed according to the normal. Thus, we will have with accuracy of a very high degree

$$\cos\gamma = r_0, \quad dF \sqrt{1 + \left(\frac{\partial x_3}{\partial x_1}\right)^2 + \left(\frac{\partial x_3}{\partial x_2}\right)^2}\, r_0\, dr_0\, d\theta$$

where $x_3 = x_3(x_1, x_2)$ is the equation of the surface F_0.

We shall designate by J_F the internal integral of the first item of formula (4.5). To obtain an evaluation we present it in the form

$$J_F = J_{F-F_0} + J_{F_0}$$

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In the right side of this formula the first integral is limited; for the second integral we can write

$$J_{F_0}\Big| < a\int_0^{2\pi} d\theta \int_0^{\delta} \frac{r_0^2}{8\sqrt{\pi^3 \nu^3}(t-\tau)^2} \exp \frac{-r_0^2}{4\nu(t-\tau)}\, dr_0 \qquad (4.7)$$

Here is a constant related to the surface F_0 and δ is the radius-vector of the contour of the surface F_0. Substitution of

$$\frac{r_0}{2\sqrt{\nu(t-\tau)}} = \omega, \qquad dr_0 = 2\sqrt{\nu(t-\tau)}\, d\omega$$

gives

$$\Big|J_{F_0}\Big| < a\sqrt{\frac{\nu}{\pi^3}} \int_0^{2\pi} d\theta \int_0^{\omega^*} \omega^3 \exp(-\omega^2)\, d\omega \qquad \left(\omega^* = \frac{\delta}{2\sqrt{\nu(t-\tau)}}\right)$$

The right part of the latter inequality is limited for all δ and t; therefore, it can be concluded that J_{F_0} is of limited value and

$$\Big|J\Big| < c\sqrt{t} \qquad (4.8)$$

Let f_0 be the maximum value among the given $|f_i|$ in system (4.1). On the basis of the determination of (4.3), by formula (4.3) we obtain

$$\Big|W_{i0}\Big| \leqslant f_0, \qquad \Big|W_{i1}\Big| < c f_0 \sqrt{t} \qquad (4.9)$$

By (4.5) and (4.8) we determine w_{i2}

$$\Big|W_{i2}\Big| < c f_0 \left| \int_0^t \frac{\sqrt{\tau}\, d\tau}{\sqrt{t-\tau}} \int_F \sqrt{\tau - t}\left(A + \sum_k V_{ki}\right) dF \right|$$

Substituting $\tau = t\omega$, we will have

$$\left| W_{i2} \right| < cf_0 (c_1 c + c_2 t) \int_0^1 \frac{\sqrt{\omega}}{\sqrt{1-\omega}} \, d\omega$$

where c_1 and c_2 are positive constants $c_1 + c_2 = c$.

Therefore, we may write

$$\left| W_{i2} \right| < c^2 t f_0 \int_0^1 \frac{\sqrt{\omega}}{\sqrt{1-\omega}} \, d\omega$$

It is known from the theory of the gamma functions that

$$\int_0^1 \frac{\omega^{\alpha} \, d\omega}{(1-\omega)^{\beta}} = \frac{\Gamma(\alpha+1)\Gamma(1-\beta)}{\Gamma(\alpha+2-\beta)} \qquad (\beta < 1)$$

On the basis of this last equation, we obtain

$$\left| W_{i2} \right| < c^2 t f_0 \, \frac{\Gamma(3/2)\,\Gamma(1/2)}{\Gamma(2)} \qquad (4.10)$$

Evaluation of the third member of the series (4.2) is carried out in an analogous manner

$$\left| W_{i3} \right| < c^3 \sqrt{t^3} f_0 \, \frac{\Gamma(3/2)[\Gamma(1/2)]^2}{\Gamma(5/2)} \qquad (4.11)$$

Continuing this process of determination, we obtain for the general term

$$\left| W_{im} \right| < c^m \sqrt{t^m} f_0 \, \frac{\Gamma(3/2)[\Gamma(1/2)]^{m-1}}{\Gamma(\frac{1}{2}m+1)} = \frac{\pi f_0 c^m \sqrt{t^m}}{2\Gamma(\frac{1}{2}m+1)} \qquad (4.12)$$

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Therefore, the ratio of the two successive terms of the majority of series (4.2) will be

$$cx\sqrt{t}\;\frac{\Gamma(\frac{1}{2}m+1)}{\Gamma[\frac{1}{2}(m+1)+1]}$$

which at all finite values of t and x tends toward zero when $m \to \infty$; consequently, with $x = 1$ series (4.2) gives a solution of system (3.8).

2. It is easily shown that the solution of system (3.8) is unique. Actually, the existence of two different continuous solutions of system (3.8) would indicate the existence of a continuous solution other than zero of the corresponding homogeneous system

$$\lambda W_i(N,t) + \int_F dF \int_0^t \sum W_k(M,\tau)\, v_{ki}(N,M,t-\tau)\, d\tau = 0 \qquad (4.13)$$

We shall designate the upper limit of the absolute values of the solution of system (4.13) by w_0. By equation (4.13) and the preceding determinations we have

$$|W_i| \leqslant W_0 J$$

or

$$|W_i| < cW_0\sqrt{t}$$

On the basis of this determination, from equation (4.13) we obtain

$$|W_i| < cW_0 t\,\frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(2)}$$

A continuation of this determination process leads us to a general determination analogous to (4.12):

$$|W_i| < \frac{\pi W_0 c^m \sqrt{t^m}}{2\,\Gamma(\frac{1}{2}m+1)}$$

From this we obtain $w_i(N, t) \to 0$ with $m \to \infty$, which was to be demonstrated.

3. The solution of the plane problem can be obtained from formulas (3.1) and (3.2), changing the indices i and k from 1 to 2 and taking the expression (2.8) in the form

$$A = \frac{r\cos\gamma}{4\pi\nu t^2} \exp\frac{-r^2}{4\nu t}, \quad V_{ik} = \frac{n_k(x_i-\xi_i)}{4\pi\nu t^2} \exp\frac{-r^2}{4\nu t} + \frac{\partial\varphi_i}{\partial x_k} \quad (4.14)$$

Considering that the fundamental solutions of the plane problem according to Odqvist have the form

$$u_{ik} = \delta_{ik}\frac{\cos\gamma}{\pi r} + U_{ik}$$

where

$$U_{ik} = \frac{n_k(x_i-\xi_i)}{\pi r^2} + \frac{\partial\Phi_i}{\partial x_k}, \quad \Phi_i = -\frac{x_i-\xi_i}{\pi r^2}\sum n_k(x_k-\xi_k)$$

we determine the function φ_i from the boundary problem

$$\Delta\varphi_i = -2\frac{\partial A}{\partial x_i}, \quad \varphi_i(N, M, t) = \frac{r^2\Phi_i(N,M)}{4\nu t^2} \exp\frac{-r^2}{4\nu r} \quad (4.15)$$

Introducing the expression

$$\psi_i(P, M, t) = \frac{r^2\Phi_i(P,M)}{4\nu\tau^2} \exp\frac{-r^2}{4\nu t}$$

we will have

$$B_i = \frac{r^2\Phi_i}{2\nu^2 t^3}\left(1 - \frac{r^2}{8\nu t}\right)\exp\frac{-r^2}{4\nu t}, \quad \int_0^t B_i\, d\tau = -\frac{r^2\Phi_i}{4\nu^2 t^2} \exp\frac{-r^2}{4\nu t}$$

Further,

$$\int_0^t V_{ik}\, d\tau = \int_0^t \left[V_{ik} - \frac{r^2 U_{ik}}{4\nu(t-\tau)^2} \exp\frac{-r^2}{4\nu(t-\tau)}\right] d\tau + U_{ik}\int_0^t \frac{r^2}{4\nu(t-\tau)^2} \exp\frac{-r^2}{4\nu(t-\tau)}\, d\tau$$

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Substituting $t - \tau = r^2 / 4\nu\omega$, we obtain

$$\left| \int_0^t \frac{r^2}{4\nu(t-\tau)^2} \exp \frac{-r^2}{4\nu(t-\tau)} \, d\tau \right| < 1$$

The question is again reduced to the study of the behavior of the integral

$$J_{ik} = \int_0^t \left[\frac{\partial \varphi_i}{\partial x_k} - \frac{r^2}{4\nu(t-\tau)^2} \frac{\partial \Phi_i}{\partial x_k} \exp \frac{-r^2}{4\nu(t-\tau)} \right] d\tau =$$

$$= \int_0^t \frac{\partial}{\partial x_k} (\varphi_i - \psi_i) \, d\tau - \frac{x_k - \xi_k}{2\nu r} \Phi_i \exp \frac{-r^2}{4\nu t} \qquad (4.16)$$

Taking into consideration that

$$\varphi_i - \psi_i = \int_D B_i(Q, M, t) G(P, Q) \, dD_Q$$

where G(P, Q) is the Green harmonic function on the plane, and replacing in formulas (3.1) and (3.2) the integrals along the surface F by the integrals along the contour C which limits the examined plane field of motion, we obtain by the known formulas a system of two integral equations

$$\lambda w_i(N, t) + \int_C ds \int_0^t \sum_k w_k(M, \tau) \nu_{ki}(N, M, t - \tau) \, d\tau = f_i(N, t) \qquad (4.17)$$

From this the unknown densities w_1 and w_2 are determined.

Equations (4.17) form a quasi-regular system of two integral equations of the Volterra type for which the above-presented conclusions on the existence and uniqueness of the solution of system (3.8) hold.

We shall indicate one characteristic difference of the principal value between the unsteady and the corresponding steady processes. While with $\lambda = -1$ (the external field), generally speaking, system (2.5) has no solution (Stokes' paradox), system (4.17) has a solution with all continuous values f_1 and f_2 both for $\lambda = +1$ and for $\lambda = -1$, even though $t < \infty$. This difference is due to the different character of the integral equations, namely, in the first case we have an equation of the Fredholm type, while in the second the equation is of the Volterra type. On the other hand, from the physical point of view such a difference, it would seem, is abnormal.

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In connection with the latter circumstance it is exceptionally interesting to examine the question of the limiting transfer with $t \to \infty$. In all probability we would have to expect that within limits we would obtain a solution of the steady problem. However, the external two-dimensional problem of hydrodynamics confirms the fact that it is clearly impossible to consider the concurrence of the limiting solution of the unsteady problem with the solution of the steady problem.

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