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AXIALLY SYMMETRICAL MAGNETOHYDRODYNAMIC EQUILIBRIUM CONFIGURATIONS

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The conditions for magnetohydrostatic equilibrium are studied in the case of axial symmetry. The magnetic field is divided into its meridional and its toroidal parts which are described by the scalar functions F and T respectively. It is shown that the gas pressure p and the functions F and T have to be functions of each other. Taking in particular p(F) and T(F) as known relations, a differential equation for F is derived. The cases in which this differential equation is linear are considered and explicitly solved if furthermore T(F) = const. In a special case, the magnetic lines of force are calculated numerically and shown in a figure. Some remarks on the stability are added.

A magnetic field exerts forces on a conducting body when electrical currents flow through the latter intersecting the lines of flux. Conducting bodies of special interest for astrophysics and for many terrestrial applications are plasma, i.e. gaseous conductors. If gravitational effects are unimportant, then a static equilibrium can exist generally only if the forces exerted by the magnetic field and the gas pressure of the plasma are compensated everywhere. Since the forces caused by pressure are rotation-free, this equilibrium generally cannot be fulfilled but corresponds to a requirement of the configuration of the magnetic field. A special case of equilibrium exists when the currents in the conductor have a flow which is parallel to the magnetic field everywhere. Such a force-free magnetic field has been discussed by us in earlier reports.[1] If the possibility of an equilibrium between magnetic forces and gas pressure is to be utilized for the collection and enclosure of a plasma by a magnetic field (for example, to avoid contact of the plasma with material walls), forces on the plasma are required, however, and the more general equilibrium problem must be investigated. We limit ourselves in this case to axially-symmetrical arrangements and consider especially solutions which can be solved strictly analytically. For a similar axially-symmetrical case in which it has been assumed that all currents flow on the surface of the plasma, a solution by a series expansion has been given in a different report.[2]

1. Equilibrium Conditions

The equilibrium between the gas pressure and the magnetic force is described by the magnetohydrostatic equation

$$\text{grad } p = -\frac{1}{4\pi} [\mathfrak{B} \text{ rot } \mathfrak{B}]. \qquad (1)$$

Here we have p as the gas pressure and $\mathfrak{B}$ as the magnetic field. Let us assume an axial symmetry in the following, i.e. all scalar functions which occur shall depend only upon the distance s from the z-axis (symmetry axis) and upon the distance z from the meridian plane but not upon the azimuth φ.

A general cylindrical-symmetrical magnetic field can be divided as shown earlier,[3] into a poloidal and toroidal part:

$$\mathfrak{B} = \frac{1}{s^2}[[\mathbf{e}_z \mathfrak{r}] \operatorname{grad} F] + \frac{1}{s^2}[\mathbf{e}_z \mathfrak{r}] T \qquad (2)$$

We have $\mathfrak{r}$ as the local vector and $\mathbf{e}_z$ as the unit vector in the z-direction. The function F(s,z) has the importance that the course of a line of flux in the meridian plane is described by the equation F(s,z) = const in such a manner that the entire flux through a circle is given by $2\pi F(\mathfrak{r})$ which is formed by the rotation of the point $\mathfrak{r}$ around the symmetry axis in case F = 0 on the symmetry axis. The function T(s,z) has the analogous importance for the lines of the electric current.

On account of the assumed axial symmetry, the following holds true for p, F and T:

$$([\mathbf{e}_z \mathfrak{r}] \operatorname{grad} p) = 0, \quad ([\mathbf{e}_z \mathfrak{r}] \operatorname{grad} F) = 0 \quad \text{and}$$

$$([\mathbf{e}_z \mathfrak{r}] \operatorname{grad} T) = 0 \qquad (3)$$

The rotation of the magnetic field is thus given by:

$$\operatorname{rot} \mathfrak{B} = \frac{1}{s^2}[\mathbf{e}_z \mathfrak{r}] \, \mathfrak{G}F - \frac{1}{s^2}[[\mathbf{e}_z \mathfrak{r}] \operatorname{grad} T] \qquad (4)$$

$\mathfrak{G}$ is a differential term in the above which is defined by[3]:

$$\mathfrak{G} = \frac{\partial}{\partial s^2} - \frac{1}{s}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial z^2}$$

As shown by Chandrasekhar,[4] $\mathfrak{G}$ is identical with the Laplace term for an axially symmetrical function in a five-dimensional euclidic space. Substitution of Equations (2) and (4) into the equilibrium Equation (1) now results in:

$$\text{grad } p = -\frac{1}{4\pi}\left\{\frac{1}{s^2}(GF)\text{ grad } F + \frac{1}{s^2} T \text{ grad } T + \frac{1}{s^2}[e_z r]\left([e_z r]\cdot[\text{grad } T \text{ grad } F]\right)\right\} \quad (5)$$

Only the last term on the right side is purely toroidal. It must disappear for the satisfaction of the equilibrium, i.e. the magnetic fields must be free from angular momentum.[3] Therefore, we must have

$$[\text{grad } T, \text{ grad } F] = 0 \quad (6)$$

This equation states that the lines F = const (meridional projections of the lines of flux) must coincide with the lines T = const (meridional projections of the lines of the electric current). It is satisfied when F and T are functions of each other (not necessarily singular or reciprocal).

Thus we have for Equation (5)

$$\text{grad } p = -\frac{1}{4\pi s^2}\left\{(GF)\text{ grad } F + \frac{1}{2}\text{ grad } T^2\right\} \quad (7)$$

Let us take T^2 as a function of F and let

$$T^2 = g(F) \quad (8)$$

Thus Equation (7) is changed to

$$\text{grad } p = -\frac{1}{4\pi s^2}\left(GF + \frac{1}{2}\frac{dg}{dF}\right)\text{ grad } F \quad (9)$$

It follows from this equation that the lines p = const and F = const must coincide, i.e. p and F are also functions of each other. From Equation (9) we finally obtain the following equation (see also Chandrasekhar and Prendergast)[5]:

$$\Delta F + \frac{1}{2}\frac{dg(F)}{dF} = 4\pi s^2 \frac{dp(F)}{dF} \qquad (10)$$

(10) is a differential equation for F if the pressure p and the toroidal magnetic field $\sqrt{g}$ have already been disposed of as functions of F.

2. Special Axially-Symmetrical Fields

In the following such formulations will be selected for p and s that the differential Equation (10) is linear for F. Therefore, we let

$$\frac{dp}{dF} = -\frac{1}{4\pi}(aF + b) \qquad (11a)$$

and

$$\frac{dg}{dF} = 2(cF + d) \qquad (11b)$$

where a, b, c and d are constants. Thus, we have for Equation (10)

$$\Delta F + cF + d = s^2(aF + b) \qquad (12)$$

The case a = b = 0 (i.e. constant pressure) leads to force-free magnetic fields. The case c = d = 0 indicates T = const, i.e. the toroidal component of the magnetic field is free from vortexes and does not exert any force. This case will be discussed in the following. We will now search for solutions of the differential equation

$$\Delta F = s^2(aF + b) \qquad (13)$$

Let us assume first that $a \neq 0$. Then a solution is given by

$$F = -\ldots a, \quad a \neq 0 \qquad (14)$$

An additive constant in F is meaningless since the magnetic field, according to Equation ( ), is determined only by derivations of F.

Since the general solution of the differential equation is given by the superimposition of the general solution of the homogeneous equation and of the above special solution of the inhomogeneous equation, we need only be interested in the following for solutions of the homogeneous equation. For F we formulate a separate equation

$$F = S(s)\, Z(z) \qquad (15)$$

This results in the two differential equations for S(s) and Z(z):

$$\frac{d^2}{ds^2}S - \frac{1}{s}\frac{d}{ds}S - (as^2 + \lambda)S = 0 \qquad (16)$$

and

$$\frac{d^2}{ds^2}Z + \lambda Z = 0 \qquad (17)$$

λ here denotes the separation constant. The general solution can be obtained by superimposition of the solutions of various λ, i.e. by integration over λ:

$$F(s,z) = \int_{-\infty}^{+\infty} S(s;\lambda)Z(z;\lambda)d \qquad (18)$$

The solution of Equation (17) is given by

$$Z(z;\lambda) = A(\lambda)e^{i\sqrt{\lambda}z} + B(\lambda)e^{-i\sqrt{\lambda}z} \qquad (19)$$

with A and B as the integration constants. When $\lambda < 0$ then we obtain solutions which depend exponentially upon z and for $\lambda > 0$ periodic solutions are obtained.

For the solution of the differential Equation (16) let us perform a transformation of variables

$$t = \sqrt{a}\, s^2, \quad a > 0 \tag{20}$$

Then we have for (16):

$$4\frac{d^2S}{dt^2} = (t + \frac{\lambda}{\sqrt{a}})S = 0 \tag{21}$$

With the transformation

$$S = te^{-t/2}y(t) \tag{22}$$

we finally arrive at a differential equation of the confluent hypergeometrical type

$$t\frac{d^2y}{dt^2} + (2 - t)\frac{dy}{dt} - (\frac{1}{4}\frac{\lambda}{\sqrt{a}} + 1)\, y = 0 \tag{23}$$

The general solution for S is thus given by:

$$S(s;\lambda) = \sqrt{a}\, s^2 e^{-(\sqrt{a}/2)s^2}\Big\{C(\lambda)F(1 + \frac{1}{4}\frac{\lambda}{\sqrt{a}}, 2, \sqrt{a}\, s^2) + D(\lambda)[F(1+\frac{1}{4}\frac{\lambda}{\sqrt{a}}, 2, \sqrt{a}\, s^2)\cdot \ln(\sqrt{a}\, s^2) + F^*(1+\frac{1}{4}\frac{\lambda}{\sqrt{a}}, 2, \sqrt{a}s^2)]\Big\} \tag{24}$$

C and D in the above are integration constants. $F(\alpha, \gamma, x)$ is the so-called confluent hypergeometrical function[6] and $F^*(\alpha, \gamma, x)$ is a potential series:

$$F^* = \frac{\alpha}{\gamma}\frac{x}{1!}(\frac{1}{\alpha} - \frac{1}{\gamma} - 1) + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}(\frac{1}{\alpha} - \frac{1}{\alpha+1} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - 1 - \frac{1}{2}) + \ldots \tag{25}$$

$S(s,\lambda)$ is regular in the null point, while it increases

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exponentially for large values of s, since $F(\alpha, \gamma, x) \sim e^x$ for large x. In order to retain a regular magnetic field in the null point, the integration constant D must be zero.

Up to the present, it had been assumed that $a \neq 0$. The case $a = 0$ will now be investigated. According to Equation (11a) this means that the gas pressure

$$p = -(b/4\pi) \cdot F + \text{const}$$

If b would remain zero, this would mean that the electrical current density ($\sim$ rot $\mathfrak{B}$) disappears. This case will be disregarded and $b \neq 0$. For the function F we now have the differential equation

$$GF = b\, s^2 \tag{26}$$

Since $(1/s)GF$ in the meridional magnetic fields under consideration here is proportional to the electrical current density, then the magnetic forces are in equilibrium with the pressure forces if the current density increases proportionally to the distance from the symmetry axis and if it is independent of z.

An inhomogeneous solution is given by:

$$F_i = \frac{b}{8} s^4 \tag{27}$$

while for the homogeneous part the separation Equation (15) results in the differential equation

$$\frac{d^2S}{ds^2} - \frac{1}{s}\frac{dS}{ds} - \lambda S = 0 \tag{28}$$

for the function S, while Equation (17) continues to be valid for Z. The differential Equation (27) is of the Bessel type and the solutions are Bessel solutions $Z_1(x)$ of an imaginary argument. The solution which is regular in the null point is given by:

$$S(s;\lambda) = C(\lambda)\, sI_1(i\sqrt{\lambda}\, s) \qquad (29)$$

C is an integration constant which is either imaginary or real, depending upon whether the value of the Bessel function is imaginary or real corresponding to the sign of λ. When $\lambda \geq 0$, then $S(s;\lambda)$ increases exponentially for large values of s, while $S(s;\lambda)$ for $\lambda < 0$ is proportional to $\sqrt{s}$ when s moves towards infinity. In this case the magnetic field moves towards infinity for large values of $|Z|$.

For $\lambda = 0$ we obtain specially:

$$S(s;0) = Ds^2 + E \qquad (30a)$$

and

$$Z(s;0) = G\, z + K \qquad (30b)$$

where D, E, G and K are integration constants.

Solutions which are periodical in z, for example, will be discussed in greater detail in the following. In this case $\lambda \geqq 0$ and the integration constant $G = 0$. Then the value of the integration constant E is negligible. Then we have for the flux function $F(s,z;\lambda)$ according to Equations (19), (27), (29), (30a) and (30b):

$$F(s,z;\lambda) = A_1\, isI_1(i\sqrt{\lambda},s)\cos(\sqrt{\lambda},z) + B_1 s^2 + \frac{b}{8} s^4 \qquad (31)$$

In the above the factor i has been chosen in such a manner that A_1 is a real integration constant. B_1 is also an integration constant. An additional free integration constant has been set equal to zero which denotes only a determination of the phase position with respect to z.

A field resulting from this function $F(s,z;\lambda)$ has been shown in Fig. 1. The parameters have been chosen in such a manner that the gas pressure

$$p = -(b/4\pi) F + \text{const}$$

is maximum on the axis and always decreases for all s in the vicinity of s = 0 for increasing s. For the special parameters of Fig. 1 ($A_1 = 1$, $B_1 = 1$, $b = 1$) it is the case for the vicinity of the axis up to the line of force on which $F \approx 9.6$. By a suitable choice of constants available in the pressure (= gas pressure on the axis), it is then possible to obtain a positive pressure everywhere in the tube thus formed and assume a given value, for example, p = 0 on an arbitrary line of flux. This line of flux can then be identified by the wall of the vessel in the interior of which the magnetic field holds the plasma entirely (for p = 0 on the wall) and partially together and there our equations are no longer valid on its exterior; in contrast to the above, the magnetic field is formed by a corresponding arrangement of coils.

Fig. 2 shows the graph of the magnetic field strength and of the gas pressure p on the lines $z = 0, \pm 2\pi, \ldots$ and

$z = \pm\pi, \pm 3\pi \ldots$ as a function of the distance from the symmetry axis. In addition, the function $(B^2/8\pi) + p$ (= "total pressure" = "magnetic pressure" + gas pressure) has been plotted also. In the case of an extended magnetic field, this function would be constant while in this case it shows the influence of curvature.

3. Stability of Axially-Symmetrical Fields

In conclusion we will mention briefly the stability of the meridional fields under consideration here. In an earlier report,[7] the stability of general equilibrium configurations has been investigated. In the case of meridional magnetic fields, we obtain the following from the cited Equation (23):

$$-\omega^2 \int \varrho\, v^2 d\tau = -\int \left\{ \gamma p (\operatorname{div} v)^2 + \frac{1}{4\pi} (\operatorname{rot} [v\, B])^2 \right\} d\tau$$
$$+ \int (v \operatorname{grad} F)^2 \frac{d^2 p(F)}{dF^2} d\tau \qquad (32)$$
$$+ \int \left\{ \frac{1}{s^2} \frac{dp}{dF} ([e_z r] v)([e_z r] \cdot \operatorname{grad} (v \operatorname{grad} F)) \right.$$
$$\left. - \frac{dp}{dF} (v \operatorname{grad} F)([e_z r] \cdot \operatorname{grad} \frac{1}{s^2} ([e_z r] v)) \right\} d\tau$$

where v is the velocity of the plasma, γ is the ratio of the specific heat and $d\tau$ is the element of volume. (For the derivation of Equation (32) it has been assumed that the normal components of v and B at the surface of the considered volume will disappear.) The stability of an equilibrium configuration is determined by the sign of ω^2, whereby $\omega^2 < 0$ denotes instability. It can be seen from Equation (32) that the first integral always results in a stable part. The second integral will also always result in a stable part provided that $d^2p(F)/dF^2 \leqq 0$ everywhere. The sign of the last term can be positive as well as negative. But it can be shown that the integral disappears if the disturbance [illegible] is independent of the azimuth φ. Meridional fields will be stable to those disturbances in case $d^2p(F)/dF^2 \leqq 0$. For the field described by Equation (31) (see Fig. 1) the Equation (11a) is also $d^2p/dF^2 = 0$. This field is thus stable to disturbances which do not depend upon φ.

We wish to express our gratitude to Mr. A. Kurau for the mathematical calculations which have been performed with the electronic calculator G2.

Footnotes

1) R. Lüst and A. Schlüter, Z. Astrophys. 34, 263 (1954).

2) L. Bierman, K. Hain, K. Jürgens and R. Lüst, Z. Naturforschg. 12a, 826 (1957).

3) R. Lüst and A. Schlüter, Z. Astrophys. 38, 190 (1955).

4) S. Chandrasekhar, Proc. Nat. Acad. Sci. 42, 1 (1956).

5) S. Chandrasekhar and K. H. Prendergast, Proc. Nat. Acad. Sci. 42, 5 (1956).

6) E. Kamke, Gewöhnliche Differentialgleichungen (Ordinary Differential Equations), Akad. Verlagsgesellschaft, Leipzig 1943, p. 427.

7) K. Hain, R. Lüst and A. Schlüter, Z. Naturforschg. 12a, 833 (1957).

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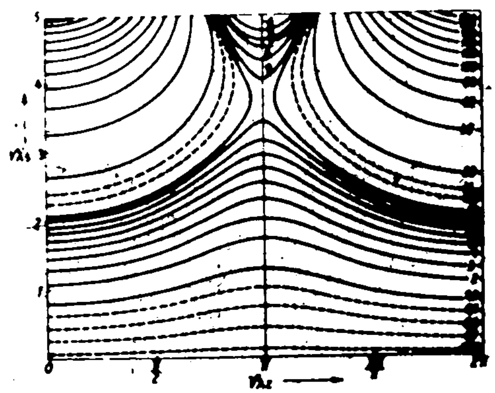


Fig. 1.--Course of the magnetic field which is defined by the flux function F according to Equation (31) with the parameter values $A_1 = B_1 = b = 1$. The numbers at the flux lines are a measure for the magnetic flux which passes through the circular cross-section between the corresponding flux line and the z-axis.

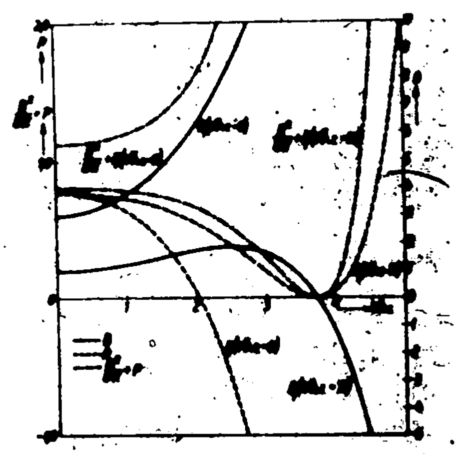
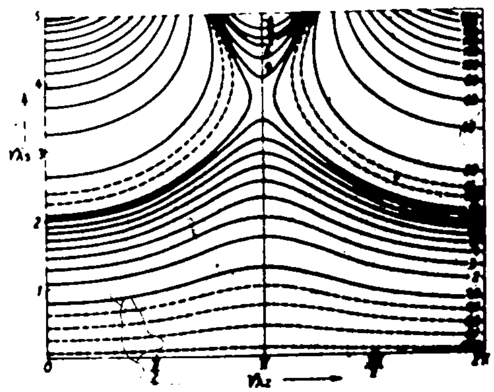


Fig. 2.--The magnetic field strength B (solid curve), the gas pressure p (dotted line) and the "total pressure" $p + B^2/8\pi$ (dot-dash curve) in relation to the distance s from the symmetry axis for $\sqrt{\lambda} z = 0, \pm 2\pi \ldots$ and for $\sqrt{\lambda} z = \pm \pi, \pm 3\pi \ldots$ —— B; ---P, -.-.- $B^2/8\pi + P$.



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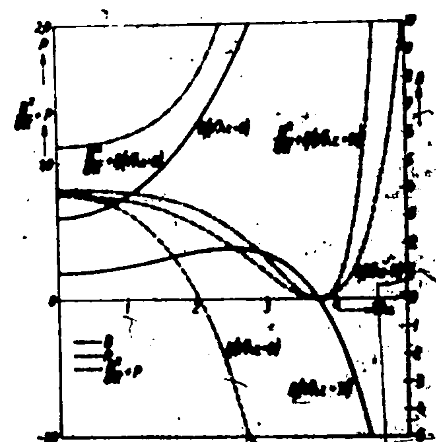

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