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THE THEORY OF THE VISCOSITY OF HELIUM II: [redacted]

Part 2: II. Calculation of the viscosity coefficient of.

by

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Summary: The kinetic equations for rotons and phonons are solved by using the effective differential cross-sections ~~for the~~ of scattering of elementary excitations (phonons and rotons) ,by each other, as obtained in the first part of this work. ~~It~~ is shown ~~that the~~ viscosity coefficient of Helium II's ~~is~~ to be composed of two parts : one caused by scattering of rotons ("roton viscosity") and not dependent on temperature; the other, caused by scattering of phonons ("phonon viscosity") and sharply increasing with fall in temperature. The experimental values for the viscosity coefficient of in helium II appear to be in good agreement with theory.

5. The kinetic equation

The expressions obtained in the first part of this work for the probability of scattering of various types of excitations by each other permit to investigate ~~investigation of~~ the temperature dependence of the viscosity coefficient of helium II. The kinetic equation for elementary excitations which we have to solve may be written in the form :

$$\frac{\partial n}{\partial t} + v \nabla n = J(n) \qquad (5.1)$$

where n = n(r,v,t) is the distribution function, v the velocity, and J(n) the collision integral, the exact form of which is deduced below for each type of interaction.

We will assume that in a fluid there is a macroscopic flow of velocity u, varying with the coordinate in such a way that the velocity gradient is so small that equilibrium is almost established in each volume element corresponding to a given value of the velocity: i.e., in each volume element the distribution functions for rotons and phonons are almost equal to their equilibrium values. Thus the distribution function n for rotons (or phonons) may be represented as the sum of the equilibrium function n_0 and a small variation δn:

$$n = n_0 + \delta n \tag{5.2}$$

Since we are seeking a stationary solution of the kinetic equation appropriate to a given macroscopic flow (constant in time), the left-hand term $\partial n / \partial t$ in the kinetic equation should be taken as equal to zero. Furthermore, we need only substitute the equilibrium function n_0 (instead of n) in the approximation term $v \nabla n$ in question, since this term already contains a very small quantity - the velcoity gradient of macroscopic flow.

In a fluid moving with velocity u, the equilibrium roton distribution function appears as :

$$n_0 = \exp\left\{ -\frac{\Delta}{kT} - \frac{(p-p_0)^2}{2\mu kT} + \frac{p \cdot u}{kT} \right\} \tag{5.3}$$

Let us choose the direction of the z axis along the velocity vector u and for the sake of simplicity let us assume that the velocity gradient is directed along ~~an~~ the axis x perpendicular to the z-axis. It is evident that the value of the viscosity coefficient required is independent of the choice of coordinate system. In calculating the first viscosity coefficient we need only consider the case where the velocity u satisfies the condition div u = 0. Let us now assume that the density and temperature are constant throughout the liquid. As a result, on the left-hand side of the kinetic equation we have :

$$v \nabla n = v_x \frac{\partial n_0}{\partial x} = n_0 \frac{P_z v_z}{kT} \frac{\partial u}{\partial x}$$

If we use a spherical system of coordinates with the polar axis lying in the z direction, the expression obtained may be rewritten in the form

$$n_0 \frac{P_0 v}{kT} \frac{\partial u}{\partial x} \cos\theta . \sin\theta . \cos\phi \qquad (5.4)$$

in which it is legitimate to write P_0 instead of P since rotons possess momenta of magnitude close to P_0. In accordance with (1.2) the velocity v is given by

$$v = \frac{\partial E}{\partial P} = \frac{P - P_0}{\mu} \qquad (5.5)$$

The final kinetic equation for rotons becomes:

$$n_0 \frac{P_0 (P - P_0)}{\mu kT} \frac{\partial u}{\partial x} \cos\theta \, \sin\theta \, \cos\phi = J(n) \qquad (5.6)$$

The kinetic equation for phonons may be written in an analogous form. In this case, however, allowance must be made for the fact that phonons obey Bose statistics, and therefore their equilibrium function for a flowing gas is

$$n_0 = \left[\exp\left(\frac{cp - u.p}{kT} \right) - 1 \right]^{-1} \qquad (5.7)$$

Carrying out similar calculations to those given above for rotons, but with the dissipation function of (5.7), we obtain the kinetic equation for phonons

$$n_0 (n_0 + 1) \frac{cp}{kT} \frac{\partial u}{\partial x} \cos\theta . \sin\theta . \cos\phi = J(n) \qquad (5.8)$$

The collision integrals on the right side of the equations (5.6) and (5.8) are dependent on the nature of the interactions of rotons with each other and with phonons in the case of equation (5.6), and on the nature of the interactions of phonons with each other and with rotons in the case of equation (5.8).

The ~~value~~ of ~~the~~ viscosity coefficient evidently does not depend on the part of the liquid considered. It is convenient to select a point at which the velocity of macroscopic motion is zero. At this point the expressions for the scattering function (5.3) and (5.7) are not dependent on velocity, and coincide with the corresponding equilibrium functions in a motionless liquid.

Due to the presence of two types of excitations in helium II we have written two kinetic equations. That portion of viscosity caused by momentum transfer by rotons will be conditionally referred to as roton viscosity; and that part due to transfer of momentum by phonons as phonon viscosity. Actually one viscosity is observed, this being equal to the sum of the roton and phonon contributions.

6. Roton viscosity.

~~Provided the processes of~~ If their scattering and absorption (See Para.4) are disregarded, the change in the number of rotons in a given phase volume will take place in two ways, namely: (a) elastic scattering of rotons by rotons (Para.4); (b) scattering of rotons by phonons (Para.3).

However, simple calculations show that down to temperatures of the order of 0.6-0.7°K the contribution to the viscosity of the scattering of rotons by phonons will be negligible compared to the elastic scattering of rotons by rotons. This is not only because the number of phonons at higher temperatures is found to be less than the number of rotons, but mainly because the roton momenta are changed very little during the scattering of rotons by phonons; accordingly the momentum flux, which determines the magnitude of the viscosity, is found to be negligible. Since at temperatures below 1°K the roton viscosity becomes negligibly small in comparison with the phonon viscosity, we conclude that it is sufficient to take account only of roton-roton scattering in the kinetic equation.

The roton kinetic equation (5.6) obtained above cannot be solved accurately. However, if we do not attempt an exact calculation of the roton portion of the viscosity, but limit ourselves to a determination of its temperature dependence, the kinetic equation (5.6) may be simplified. For then it is permissible to replace the collision integral on the right-hand side of this equation by :

$$-\delta n/t, \tag{6.1}$$

where t differs from the mean time between roton collisions (t_r) by a factor which is independent of temperature and is of the order of unity. Because the time interval t_r found in Section 4 contains the unknown constant V_0, we will from now simply write t_r everywhere instead of t, including the factor of order unity indicated above within the quantity V_0.

Thus the kinetic equation (5.6) becomes

$$\frac{p_0(p-p_0)}{\mu kT}\, n_0 \frac{\partial u}{\partial x} \cos\theta \sin\theta \cos\phi = -\frac{\delta n}{t_r} \tag{6.2}$$

From this we find the deviation of the distribution function δn from its equilibrium value

$$\delta n = - \frac{t_r P (P - P_0)}{\mu k T} n_0 \cos\theta . \sin\theta . \cos\phi \frac{\partial u}{\partial x} \quad (6.3)$$

We will now calculate the momentum flux. In our case only the z-component of the momentum will differ from zero, being directed towards x and equal to

$$\sigma_{xz} = - \int P_z v_x \delta n \frac{dP}{(2\pi h)^3}$$

$$= \frac{t_r}{kT\mu^2 (2\pi h)^3} \frac{\partial u}{\partial x} \int P_0^2 (P - P_0)^2 \cos^2\theta . \sin^2\theta . \cos^2\phi . n_0 \, dP . \quad (6.4)$$

Comparing (6.4) with the expression for the corresponding tensor component and pressure containing the viscosity coefficient* η_r

$$\sigma_{xz} = \eta_r \frac{\partial u}{\partial x}$$

we find the ~~magnitude~~ of viscosity coefficient required:

$$\eta_r = \frac{t_r}{kT\mu^2 (2\pi h)^3} \int P_0^2 (P - P_0)^2 \cos^2\theta . \sin^2\theta . \cos^2\phi \, n_0 \, dP . \quad (6.5)$$

We carry out the necessary integration in (6.5) over the elements of phase volume dP using the distribution function n_0 given by (5.3). The final result is

$$\eta_r = (t_r P_0^2 / 15\mu) N_r = J_r P_0^2 / 15\mu \quad (6.6)$$

The quantity J_r, equal to the product $t_r N_r$ is independent of temperature. And actually, according to (4.11), we have

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*The suffix r is used to distinguish the roton part of the viscosity coefficient.

$$J_r = t_r N_r = h^4 / P_0 \mu |V_0|^2 \qquad (6.7)$$

Thus the roton portion of the (viscosity) coefficient of is given in terms of constant quantities and is therefore a constant independent of temperature.

7. Establishment of equilibrium in ~~of~~ a phonon gas.

A change in the number of phonons in ~~the~~ an element of phase volume ~~element~~ may occur in the following ways: (a) through scattering of phonons by phonons (Section 2); (b) by absorption and emission of phonons due to inelastic ~~impacts~~ collisions of rotons or phonons with each other.

In considering the role of each of these processes in the phenomena of viscosity, we first calculate the relaxation times (or the corresponding free paths) characterizing the establishment of equilibrium in a phonon gas.

In Section 2 it was shown that the effective cross-section of phonon-phonon scattering reaches its maximum importance for small angles between the momenta of the colliding phonons. It follows from the laws of conservation of momentum and energy that such a process of scattering does not lead to a material change in the direction of the momenta of colliding phonons, and therefore scattering of phonons by phonons leads mainly to rapid exchange of energies between phonons. However, an exact calculation of the relaxation time characteristic of the establishment of energy equilibrium in a phonon gas cannot be accomplished for the simple reason that the problem itself cannot be precisely formulated. Nevertheless, since the time required for establishment of energy equilibrium in a phonon gas is an extremely important characteristic of our system, we will attempt to approach its determination from two limiting cases.

In the first case we assume that by some means there has been produced in the phonon gas a change in the number of phonons possessing small energies (smaller than the average phonon energy) so that the distribution function for the region of small energies does not equal the equilibrium function; then we calculate the relaxation interval characterizing the establishment of equilibrium in such a gas.

In the second limiting case we assume that in the phonon gas there has been a change in the number of phonons having large energies (greater than the average energy of the phonons, so that the distribution function in the region of large energies does not equal the equilibrium function. In this case we calculate the relaxation time characterizing the establishment of equilibrium in such a gas. Comparing it with the time characterizing the viscous processes of transfer, we show that the process of establishing energ~~etic~~y equilibrium in a phonon gas is more rapid than the viscous processes.

We start with the first case; that is, we examine the scattering of small-energy phonons by phonons. In this case it may be assumed that the momentum p of the phonon under question is considerably smaller than the momentum p_1 of the phonon with which this phonon collides. As was shown in Section 2, such an assumption considerably simplifies the expression for the averaging-over of angles of ~~the~~ phonon-phonon scattering. In the case in question, the angularly normalized effective cross-section of phonon-phonon scattering is by (2.11):

$$\sigma(p, p_1) = \frac{\pi (u+2)^4 p_1^4}{(96 \pi h^2 \rho_0 c)^2 \gamma}.$$

Let us look at the kinetic equation for phonons

$$\frac{\partial n}{\partial t} + v \nabla n = J(n)$$

where the collision integral J(n) for the process of phonon-phonon scattering equals

$$J(n) = -(2\pi h)^{-3} \int c\, \sigma(p, p_1) \left\{ n n_1 (n'+1)(n_1'+1) - n' n_1' (n+1)(n_1+1) \right\} d\underline{p}_1 \qquad (7.1)$$

We are interested in the relaxation of small-energy phonons, whose distribution function equals n, with the given equilibrium distribution of the remaining phonons. Therefore the distribution functions n_1, n' and n_1' in (7.1) will be regarded as in equilibrium, and the deviation of the distribution function n from the equilibrium value n_0 will be regarded as small and equal to δn. Now utilizing certain properties of equilibrium distribution functions, we transform the expression enclosed in parenthesis in (7.1)

$$\left\{ (n_0 + \delta n) n_1 (n'+1)(n_1'+1) - n' n_1' (n_1+1)(n_0 + 1 + \delta n) \right. \qquad (7.2)$$
$$= \delta n \left\{ n_1 (n'+1)(n_1'+1) - n' n_1' (n_1+1) \right\} = \delta n\, n' n_1' (n_1+1) / n$$

Substituting the expression obtained in the collision integral, we have

$$J(n) = -(2\pi h)^{-3} \delta n \int n^{-1} n' n_1' (n_1+1)\, c\, \sigma(p, p_1)\, dp_1 \qquad (7.3)$$

Thus the relaxation time for phonons with small energy is determined by the relationship

$$1/t_M = (2\pi h)^{-3} \int n^{-1} n' n'_1 (n_1 + 1) \, c \, \sigma(p, p_1) \, dp_1 \qquad (7.4)$$

In order to simplify the integration of (7.4), let us substitute the functions n_1, n', and n'_1 by the Wien function and make allowance for the fact that the momenta p and p_1 satisfy the inequality $p \ll p_1$. In this way the combination of distribution functions entering into expression (7.4) appears as: $n_1 (n_1 + 1) pc/kT$. Using this and substituting in (7.4) expression (2.11) for $\sigma(p, p_1)$, we have*

$$\frac{1}{t_M} = (2\pi h)^{-3} \frac{\pi (u+2)^4}{(96\pi h^2 \rho_0)^2 c \gamma} \int n_1 (n_1 + 1) p_1^4 dp_1 \frac{pc}{kT}. \qquad (7.5)$$

or after an elementary integration

$$\frac{1}{t_M} = \frac{(u+2)^4 \, 6!}{(48 h^2 \rho_0)^2 c \gamma (2\pi h)^3} \left(\frac{kT}{c}\right)^6 p. \qquad (7.6)$$

Automatic allowance is made for the fact that phonons having momentum p_1 possess high energy, since in the integration over dp_1 a substantial role is played by phonons with energy of the order of 6 kT.

Let us procede to the case of relaxation of high energy phonons. In this case the assumption that the phonon momenta obey the relation :

$$p_1 \lll p \qquad (7.7)$$

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*Integration over dp_1 with a zero lower limit results in a certain error, which however is not material, since our results are of approximate nature.

permits a simplification of the expression for the angularly normalized probability of phonon scattering by a phonon. Substituting p_1 for p in expression (2.11), we obtain

$$\sigma(p, p_1) = \frac{\pi(u+2)^4 p^4}{(96\pi\hbar^2\rho_0 c)^2\gamma} \tag{7.8}$$

In quite the same way as in the case of low-energy phonons, we find the relationship for the relaxation time t_σ for high-energy phonons

$$1/t_\sigma = (2\pi\hbar)^{-3}\int n^{-1}(n_1+1)\,n'n_1'\,c\,\sigma(p, p_1)\,d\underline{p}_1 \tag{7.9}$$

which coincides with expression (7.4). This agreement is natural, since in deriving equation (7.4) the assumption about the magnitude of the energy of relaxing phonons was nowhere employed. Let us now make use of the assumption about the smallness of the momentum; taking the phonon distribution functions n, n', and n_1' as small compared to unity, we may simplify the combination of distribution functions entering into expression (7.9). We have*

$$n^{-1}n'n_1'(n_1+1) \simeq n_1(n_1+1) \tag{7.10}$$

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Substituting (7.8) and (7.10) in equation (7.9), we have

$$\frac{1}{t_\sigma} = (2\pi\hbar)^{-2}\,\frac{\pi(u+2)^4}{(96\pi\hbar^2\rho_0)^2 c\gamma}\,p^4\int n_1(n_1+1)\,d\underline{p}_1 \tag{7.11}$$

Finally, after an elementary integration, we find

$$\frac{1}{t_\sigma} = \frac{(u+2)^4(kT/c)^3(\pi^2/3)p^4}{(48\hbar^2\rho_0)^2 c\gamma\,(2\pi\hbar)^3} \tag{7.12}$$

The fact that the momentum p_1 is small is automatically

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*The assumption of the smallness of functions n, n', and n_1' is equivalent to the condition $p, p', p_1' < kT/c$.

satisfied in expression (7.11), because a substantial role is played in the integral contained there by energies of the order of 2KT. Using the numerical values of the parameters entering into formulae (7.6) and (7.12), we calculate the coefficients in these expressions. The result obtained is the relaxation time of low-energy phonons:

$$1/t_{n} \simeq 2 \cdot 10^{7} T^{7} x \qquad (7.13)$$

and for phonons with high energy:

$$1/t_{\sigma} \simeq 10^{5} T^{7} x^{4} \qquad (7.14)$$

Here the quantity x is related to the phonon energy by $\varepsilon = xkT$.

In determining the relaxation time of phonons possessing an arbitrary energy, an interpolation formula can be selected which would go over into equations (7.13) and (7.14) obtained in the two limiting cases. The following formula satisfies this condition:

$$1/t_{\phi} = 10^{5} T^{7} x (x+6)^{3} \qquad (7.15)$$

Scattering of phonons by phonons does not cause a substantial change in the direction of the momenta of the scattered phonons, but on the other hand, as already noted, this process provides for the establishment of equilibrium of energy in a phonon gas. The reciprocal of the calculated relaxation time $(1/t_{\phi})$ characterizes the speed with which their equilibrium is established in a phonon gas.

8. Viscosity due to the scattering of phonons by rotons and of phonons by phonons.

The effective cross-section of various scattering processes enter into the collision integral additively, and therefore we rewrite the kinetic equation for phonons (5.8) as :

$$n_0(n_0+1)\frac{cp}{KT}\frac{\partial u}{\partial x}\cos\theta\sin\theta\cos\phi = J_{I}(n) + J_{II}(n) + J_{III}(n) \quad (8.1)$$

where $J_I(n)$ is the collision integral for the process of scattering of phonons by rotons, $J_{II}(n)$ is the collision integral for the process of scattering of phonons by phonons, and finally, $J_{III}(n)$ is the collision integral associated with the process of small angle scattering of phonons by one other, as discussed in Section 7.

The kinetic equation (8.1) as written determines the distribution function for phonons of arbitrary energy moving in a given direction. Let us make an assumption, the correctness of which will be convincingly shown below. Let us assume that the process of establishing energy equilibrium in a phonon gas takes place noticeably faster than the scattering of phonons by rotons and of phonons by phonons. Then the phonons moving in a given direction will be distributed in energy in just the same way as they would in complete equilibrium in the absence of a gradient of the macroscopic velocity. According to Section 7, establishment of energy equilibrium in a phonon gas is accomplished by the process of scattering of phonons by phonons at small angles. In such a process, the directions of motion of colliding phonons and scattered phonons coincide. At each scattering process, the energy of the colliding phonons is conserved. Inasmuch as the phonons themselves do not change their direction of motion, it may therefore be asserted that the total energy of the phonons moving in a given direction remains unchanged.

Let us integrate the left and right sides of equation (8.1) over the total range of phonon energies. As a result of such an operation, the integral over all energies derived from $J_{II}(n)$ turns out to be identically zero, because the total energy of the phonons, moving in a given direction during small-angle phonon scattering, is conserved. Thus after the integration indicated, equation (8.1) becomes:

$$\frac{c}{kT}\frac{\partial u}{\partial x}\cos\vartheta.\sin\vartheta.\cos\varphi\int_0^\infty n_0(n_0+1)\varepsilon p^3\,dp = \int_0^\infty\left\{J_I(n)+J_{II}(n)\right\}\varepsilon p^3\,dp \quad (8.2)$$

(where $\varepsilon = cp$).

~~A change in~~ the total number of phonons can ~~take place~~ vary as a result of emission and absorption of phonons. The simplest of such processes is the triple-phonon one in which one phonon turns into two, or the reverse; this is forbidden because of the impossibility of simultaneous fulfillment of two conservation laws, those of energy and momentum. The prohibition indicated does not apply to the five-phonon process, i.e. for the process of absorption or emission of a phonon when two phonons collide with each other. The calculation of the probability of such a process is theoretically possible but involves extremely cumbersome calculations. It is sufficient to mention that the number of intermediate states for the transfer of two phonons into three amounts to over 50. Fortunately, it is possible to avoid these calculations, since the relaxation time corresponding to the five-phonon process can be calculated directly from experimental values for the absorption coefficient of sound in helium II.

These values show that at temperatures above 0.9°K the five-phonon process takes place more slowly than the processes of scattering characterizing viscosity. At lower temperatures, the picture is the opposite - the process of emission (or absorption) of the third phonon during collision of two phonons becomes more rapid than the processes of scattering characterizing viscosity. This circumstance is likewise confirmed by estimates of the probability of the five-phonon process.

We estimated the probability of emission or absorption of phonons during collisions of rotons among themselves*. These processes turn out to be slower than the processes of scattering characterizing viscosity for all temperatures below the lambda-point. Let us first examine the region above 0.9°K. From the above it follows that for the times characterizing viscous effects at these temperatures, the total number of phonons in motion in a given direction may be materially altered.

Nevertheless, ~~the change in~~ the total number of phonons in motion in a given direction does not ~~take place~~ change during scattering of phonons by phonons ~~at~~ for small angles. For this reason, in integrating the left and right sides of equation (8.1) for all phonons moving in a given direction, we find that the corresponding integral from $J_{III}(n)$ becomes identically zero. As a result we have

$$\frac{c}{kT}\frac{\partial u}{\partial x}\cos\theta\sin\theta.\cos\phi\int_0^\infty n_0(n_0+1)p^3dp=\int_0^\infty\{J_I(n)+J_{II}(n)\}p^3dp \quad (8.3)$$

From what has been said above about the relative slowness of the processes of emission and absorption of phonons, it follows from equation (8.3) that the change in the total number of phonons moving in a given direction is entirely dependent on the processes of scattering characterizing viscosity.

In accordance with our assumption for the times characteriz- the viscous process, the energ~~etic~~y equilibrium for phonons moving

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*This question will be more fully considered in another work.

in a given direction is entirely dependent on the processes of scattering characterizing viscosity.

In accordance with our assumption for the times characterizing the viscous process, the energ~~etic~~y equilibrium for phonons moving in a given direction is not destroyed. Thus phonons moving in a given direction have some temperature T', in general not equal to the average phonon gas temperature, and are characterized by an equilibrium distribution function appropriate to this temperature.

Nevertheless, because the number of phonons may change during the intervals characterizing viscosity, the distribution function for rotons moving in a given direction will not be simply the Planck function, but will contain some function α' similar to a chemical potential~~,~~ and dependent on the direction of the phonons' motion. Accordingly, at the temperatures under discussion (above 0.9°K) phonons moving in a given direction are described by the distribution function

$$n = \left[e^{a' + (pc/kT')} - 1 \right]^{-1} \tag{8.4}$$

where α' and T' are functions of the direction of the momentum of the phonon in question in the system of coordinates selected.

The function α' may be of either sign, and for this reason the expression $\alpha' + (pc/kT')$ will become zero ~~at~~ for certain phonon energies. However, the magnitude α', being proportional to the macroscopic velocity gradient, is extremely minute; therefore, this possibility could be realized only for phonons with extremely minute energy $\varepsilon \ll kT$. But the number of such phonons is so minute that they play no role at all in processes of scattering characteristic of viscosity; for this reason the circumstance mentioned has no influence on further calculations.

Let the average temperatures of the phonon gas be T; then the deviation of the distribution function n from the equilibrium value n_0 is, in accordance with (8.4) and (5.7),

$$\delta n = n - n_0 = -n_0 (n_0 + 1) \left\{ \alpha' - \frac{pc}{kT} \frac{\delta T}{T} \right\} \qquad (8.5)$$

where δT is the deviation in temperature of phonons moving in a given direction, from the average temperature T ($\delta T = T' - T$).

The deviation δn of the distribution function from its equilibrium value in the temperature region under consideration depends on the two variables α' and δT; to determine them we use equations (8.2) and (8.3).

We will now consider the temperature region below 0.9°K. In this case the processes of emission and absorption of phonons take place faster than the processes of scattering characteristic of viscosity. Because of the noticeable difference in the laws of temperature variation for the processes mentioned, the once comparatively slow porcesses become comparatively rapid on transition through a small interval of temperatures, so that by 0.8°K the processes of emission and absorption of phonons may already be considered rapid. Thus in the region of temperatures below 0.8°K the number of phonons moving in a given direction will be described merely by Planck's function, dependent only on the temperature T' of the phonons moving in that direction:

$$n = \left[e^{pc/kT'} - 1 \right]^{-1} \qquad (8.6)$$

In this case we obtain the following expression for the deviation of the distribution function of phonons from the equilibrium value

$$\delta n = n - n_0 = n_0(n_0+1)\frac{pc}{kT}\;\frac{\delta T}{T} \qquad (8.7)$$

We will use equation (8.2) to determine δn in this region of temperatures. Equation (8.3) has no significance here because the motion of the total number of phonons moving in a given direction is not determined by viscous processes but by the more rapid processes of emission and absorption of the phonons.

Let us calculate the phonon portion of the viscosity coefficient for the temperature region above 0.9°K. To do this equations (8.2) and (8.3) must be solved. Let us begin by a transformation of the collision integral $J_I(n)$ contained on the right-hand side of (8.2) and (8.3). The collision integral $J_I(n)$ for the scattering process of a phonon by rotons equals

$$J_I(n) = -(2\pi\hbar)^{-3}\iint c\,d\sigma(p,\psi)\left\{Nn(N'+1)(n'+1) - N'n'(N+1)(n+1)\right\}d\underline{P} \qquad (8.8)$$

Here N is the distribution function of rotons, n is the distribution function of phonons, $\underline{P}$ is the roton momentum and $\underline{p}$ is the phonon momentum. The functions for colliding phonons and rotons are written without a prime sign and the corresponding functions for scattered particles are indicated with a prime. The effective differential cross-section $d\sigma$ of the scattering of phonons by rotons ~~at~~ for an angle ψ equals, by (3.20)

$$d\sigma(p,\psi) = \left(\frac{P_0 p^2}{4\pi\hbar^2\rho_0 c}\right)^2\left\{\frac{2}{3}(1+\cos\psi)\cos^2\psi + \frac{1}{105}\left(\frac{P_0}{\mu c}\right)^2\left(1+8\cos^2\psi+\frac{8}{3}\cos^4\psi\right)\right.$$
$$\left. + \frac{2A}{15}\left(\frac{P_0}{\mu c}\right)(1+2\cos^2\psi)+A^2\right\}dO$$

(dO is an element of solid angle).

Where there is a macroscopic velocity gradient u, the distribution of rotons and phonons is characterized by non-equilibrium functions. However, it may easily be shown that due to the rapidity of the process of scattering of rotons by rotons, the deviation of their distribution functions from the equilibrium values will be considerably less than the corresponding deviations for phonons. For this reason rotons will be described by their equilibrium distribution functions in the collisions integral $J_I(n)$. For phonons, however, we will have to substitute non-equilibrium functions as in (5.2). After such substitution and integration over the elements of phase volume $(d\underline{P})$ of the colliding rotons, the collisions integral $J_I(n)$ appears as follows :

$$J_I(n) = -N_r \int c\, d\sigma(\rho, \psi) \{\delta n - \delta n'\} \tag{8.9}$$

where N_r represents the number of rotons per unit volume, and the deviation δn is determined by the expressions in the relation (8.5).

The left side of equation (8.2) and (8.3) contains a function of the angles of the form $P_{21}(\cos\theta)\ \cos\phi$, where $P_{21}(\cos\theta)$ is the associated function of Legendre. Therefore equations (8.2) and (8.3) should be solved in such a way that the expressions for the deviation δn entering into the right sides of expressions (8.2) and (8.3) should be proportional to similar functions of the angles. In the future, however, in order to simplify transformations of the collision integrals, we will use in the intermediate calculations the Legendre functions of zero order in $P_2(\cos\theta)$, instead of the function $P_{21}(\cos\theta)\cos\phi$.

Such a substitution is possible because the spherical symmetry of the scattering makes the collisions integral insensitive to it. Naturally in the final results we will substitute instead of the function $P_2(\cos\theta)$ the function $P_{21}(\cos\theta)\ \cos\phi$ obtained in the system of coordinates selected .

Thus we write the functions α' and $\delta T/T$ determining the deviation of the distribution for phonons moving in a given direction from the equilibrium value in the form

$$\alpha' = \alpha P_2(\cos\theta) \quad , \quad \frac{\delta T}{T} = \beta P_2(\cos\theta) \tag{8.10}$$

where α and β are coefficients independent of the angles. With the aid of (8.5) and (8.10) the collision integral J_I (n) becomes:

$$J_1(n) = N_r \int n_0(n_0+1)\, c\, d\sigma(p,\psi)\left(\alpha - \frac{pc}{kT}\beta\right)\left[P_2(\cos\theta) - P_2(\cos\theta')\right] \tag{8.11}$$

Let us now use the following result of Legendre's theorem on the addition of these functions. Let there be two directions in space fixed by their polar angles θ and ψ and aximuths ϕ and ϕ'; then the Legendre function of zero order (depending on the cosine of angle θ' between these directions) satisfies the following integral relationship:

$$\int_0^{2\pi} P_n(\cos\theta')\, d\phi = 2\pi\, \overset{\cos\theta}{P_n(\cos\theta)}\, \overset{\cos\psi}{P_n(\cos\psi)} \tag{8.12}$$

Expression (8.11) contains an integration over the solid angle of the scattered phonons, and therefore applying relationship (8.12) we rewrite the collision integral (8.11) as :

$$J_I(n) = P_2(\cos\theta)\, N_r \int c\, d\sigma(p,\psi)\left[1 - P_2(\cos\psi)\right]\left(\alpha - \frac{pc}{kT}\beta\right) n_0(n_0+1)$$

From this, after integration over the elements of solid angle (dO) of the scattered phonons, we obtain:

$$J_I(n) = \frac{1}{\tau} P_2(\cos\theta)\, n_0(n_0+1)\left(\alpha - \frac{pc}{kT}\beta\right)\left(\frac{pc}{kT}\right)^4 \tag{8.13}$$

where we introduce the symbol

$$\frac{1}{\tau} = \frac{N_r}{4\pi c}\left[\frac{P_0\,(kT/c)^2}{h^2 \rho_0}\right]^2\left[\frac{2}{15} + \frac{33}{35^2}\left(\frac{P_0}{\mu c}\right)^2 + \frac{14A}{75}\left(\frac{P_0}{\mu c}\right) + A^2\right] \tag{8.14}$$

The quantity τ has the dimension of time.

Next we calculate the integrals on the right-hand sides of (8.2) and (8.3), $\int_0^\infty J_I(n)\,p^2\,dp$ and $\int_0^\infty J_I(n)\,\varepsilon p^2\,dp$. The required integration is carried out very simply by taking as the distribution function n_0 the equilibrium function of Planck. Thus we find

$$\int_0^\infty J_I(n)\,p^2\,dp = P_2(\cos\theta)\left(6!\,\alpha - 7!\,\beta\right)\left(kT/c\right)^3 \frac{1}{\gamma} \tag{8.15}$$

$$\int_0^\infty J_I(n)\,\varepsilon p^2\,dp = P_2(\cos\theta)\left(7!\,\alpha - 8!\,\beta\right)\left(kT/c\right)^3 \frac{kT}{\gamma} \tag{8.16}$$

We now carry out a transformation of the collision integral J_{II} analagous to that given above, taking into consideration the process of scattering of phonons by phonons. The integral concerned is written as follows:

$$J_{II}(n) = -(2\pi h)^{-3}\iint c\,d\sigma\left(\underline{p},\underline{p}_1,\underline{p}',\underline{p}_1'\right)\left\{n n_1 (n'+1)(n_1'+1) - n' n_1' (n+1)(n_1+1)\right\} d\underline{p}_1 \tag{8.17}$$

where the differential effective cross-section $d\sigma(\underline{p},\underline{p}_1,\underline{p}',\underline{p}'_1)$ is determined by the relation (2.9).

Let us substitute in (8.17) the phonon distribution functions in the form of the sums of the equilibrium functions and small additions such as those of (8.5). Evaluation of (8.10) then gives

$$\begin{aligned} J_{II}(n) = {} & \alpha(2\pi h)^{-3}\iint c\,d\sigma\left(\underline{p},\underline{p}_1,\underline{p}',\underline{p}_1'\right) n_0 n_{10}(n_0'+1)(n_{10}'+1) \\ & \left\{P_2(\cos\theta) + P_2(\cos\theta_1) - P_2(\cos\theta') - P_2(\cos\theta_1')\right\} d\underline{p}_1 \\ & - (2\pi h)^{-3}\beta(c/kT)\iint c\,d\sigma\left(\underline{p},\underline{p}_1,\underline{p}',\underline{p}_1'\right) n_0 n_{10}(n_0'+1)(n_{10}'+1) \\ & \left\{p P_2(\cos\theta) + p_1 P_2(\cos\theta_1) - p' P_2(\cos\theta') - p_1' P_2(\cos\theta_1')\right\} d\underline{p}_1 \end{aligned} \tag{8.18}$$

Here $\underline{p}$ and $\underline{p}_1$ are the momenta of the colliding phonons, $\underline{p}'$ and $\underline{p}'_1$ are the momenta of scattered phonons, θ is the angle formed by the path of the phonon with the corresponding index and that of the phonon with the given direction. We introduce angles ψ_1, ψ' and ψ'_1 between the directions of the momenta $\mathbf{p}_1$, $\mathbf{p}'$ and $\mathbf{p}'_1$ and the momentum $\mathbf{p}$, and apply the result (8.12) of the addition theorem for Legendre functions. As a result (8.18) becomes:

$$J_D(n) = P_2(\cos\theta)\,\alpha(2\pi\hbar)^{-3}\iint c\,d\sigma(\mathbf{p},\mathbf{p}_1,\mathbf{p}',\mathbf{p}'_1)\,n_0 n_{10}(n'_0+1)(n'_{10}+1)\times$$
$$\{1 + P_2(\cos\psi_1) - P_2(\cos\psi') - P_2(\cos\psi'_1)\}\,d\mathbf{p}_1$$
$$-P_2(\cos\theta)\,\beta\frac{c}{kT}(2\pi\hbar)^{-3}\iint c\,d\sigma(\mathbf{p},\mathbf{p}_1,\mathbf{p}',\mathbf{p}'_1)\,n_0 n_{10}(n'_0+1)(n'_{10}+1)\times$$
$$\{p + p_1 P_2(\cos\psi_1) - p' P_2(\cos\psi') - p'_1 P_2(\cos\psi'_1)\}\,d\mathbf{p}_1 \qquad (8.19)$$

Let us show that the second integral in (8.19) appears even without allowance for scattering. Making the obvious substitution for $P_2(\cos\psi)$, the bracketed expression in the integral may be written as follows:

$$\frac{3}{2}\left(p'\sin^2\psi' + p'_1\sin^2\psi'_1 - p_1\sin^2\psi_1\right) \qquad (8.20)$$

We deduct from the equation for conservation of energy $p+p_1-p'-p'_1=0$, the projection of the equation of momentum conservation in the direction of the momentum $\mathbf{p}$

$$p + p_1\cos\psi_1 - p'\cos\psi' - p'_1\cos\psi'_1 = 0$$

Our result is

$$p_1(1-\cos\psi_1) - p'(1-\cos\psi') - p'_1(1-\cos\psi'_1) = 0 \qquad (8.21)$$

Multiplying the expression obtained by 3 and adding the result to expression (8.20), we obtain

$$\frac{3}{2}\left\{p_1(1-\cos\psi_1)^2 - p'(1-\cos\psi')^2 - p'_1(1-\cos\psi'_1)^2\right\} \qquad (8.22)$$

It is easy to see from equation (8.21) that the expression (8.22) is proportional to $(1-\cos\psi_1)^2$ when the angles between the momenta of colliding phonons are small. But in accordance with (2.7) the differential effective cross-section for the scattering process in question diverges in this case, if scattering varying [illegible] ~~at~~ $(1-\cos\psi_1)^2$ is disregarded. Thus the second integral in (8.19) remains finite for all angles between the momenta of colliding phonons.

If scattering is neglected the first integral in this same expression (8.19) diverges logarithmically for small angles between the momenta of colliding phonons.

The calculation of the integrals $\int_0^\infty J_{II}(n)p^2dp$ and $\int_0^\infty J_{II}(n)\varepsilon p^2dp$, which we will not reproduce here due to its extreme unwieldiness, shows that at temperatures above 0.9°K these integrals may be neglected in comparison with the corresponding integrals on the right sides of equations (8.2) and (8.3) involving $J_I(n)$. Equality of the corresponding integrals involving $J_I(n)$ and $J_{II}(n)$ occurs only at 0.7°K. Therefore when considering the phenomenon of viscosity in the temperature region above 0.9°K, we will make allowance on the right hand sides of the equations only for effects due to scattering of phonons by rotons, these being described by the collision integral $J_I(n)$.

As shown above, ~~at~~ for lower temperatures, the parameter α may be assumed equal to zero. Equation (8.2) is used to determine the only unknown parameter β.

According to (8.19), in this case integral $J_{II}(n)$ simply equals

$$J_{\text{II}}(n) = -P_2(\cos\vartheta)\,\frac{c\beta}{kT}(2\pi h)^{-3}\iint c\,d\sigma(p, p_1, p', p_1')\,n_0 n_{10}(n_0'+1)(n_{10}'+1)$$
$$\times\{p + p_1 P_2(\cos\psi) - p' P_2(\cos\psi') - p_1' P_2(\cos\psi_1')\}\,dp_1 \qquad (8.23)$$

The calculation of the integral $\int_0^\infty J_{\text{II}}(n)\varepsilon p^2 dp$, appearing on the right-hand side of equation (8.2), which we will not discuss here, gives the following result:

$$\int_0^\infty J_{\text{II}}(n)\varepsilon p^2\,dp = -P_2(\cos\vartheta)\,\beta(kT)\left(\frac{kT}{c}\right)^3 \frac{1}{\mathcal{T}_\phi} \qquad (8.24)$$

where $\mathcal{T}_\phi$ is a magnitude with the dimension of time being determined by the relationship:

$$\frac{1}{\mathcal{T}_\phi} = \frac{3.\,\underline{|13}\,(u+2)^4}{5\cdot 2^{19}(2\pi)^3 h^7 \rho_0^2 c}\left(\frac{kT}{c}\right)^9 \qquad (8.25)$$

In view of the complete convergence of the integral (8.23), expression (8.25) naturally does not contain the scattering parameter.

The integrals contained on the left-hand sides, if the equations (8.2) and (8.3) are calculated by elementary methods, and their values are

$$\int_0^\infty n_0(n_0+1)\,\varepsilon p^3\,dp = \frac{4}{15}\,\pi^4\,kT\left\{\frac{kT}{c}\right\}^4 \qquad (8.26)$$

$$\int_0^\infty n_0(n_0+1)\,p^3\,dp \simeq \frac{36}{5}\left(\frac{kT}{c}\right)^4 \qquad (8.27)$$

We now substitute in equation (8.2) and (8.3) the values for all the integrals involved. Eliminating immaterial factors and dropping angular functions, we obtain the following pair of equations for the determination of the parameters α and β valid in the temperature region above 0.9°K

$$\frac{\partial u}{\partial x}\,\frac{4\pi^4}{15} = (7!\,\alpha - 8!\,\beta)\frac{1}{\mathcal{T}}\,, \qquad \frac{\partial u}{\partial x}\,\frac{36}{5} = (6!\,\alpha - 7!\,\beta)\frac{1}{\mathcal{T}} \qquad (8.28)$$

For temperatures below 0.9°K, by applying the same method to equation (8.2) we obtain the equation for the determination of parameter β:

$$\frac{4\pi^4}{15}\frac{\partial u}{\partial x} = -\beta\left(\frac{8!}{\tau} + \frac{1}{\tau_\phi}\right) \qquad (8.29)$$

9. Length of the mean free path characterizing viscosity.

We will now calculate the length of the mean free path characterizing viscosity. The parameters α and β, which determine the deviation of the distribution function from its equilibrium value, are characterized to a considerable extent by their equilibrium values. With simultaneous change in the values of these parameters, the viscosity coefficient changes correspondingly. We will consider α and β as functions of the time t. Then in complete analogy to equation (8.28) we may obtain two equations

$$\begin{aligned} \frac{\pi^3}{3}\frac{d\alpha}{dt} - \frac{36}{5}\frac{d\beta}{dt} &= -(6!\alpha - 7!\beta)\frac{1}{\tau} \\ \frac{36}{5}\frac{d\alpha}{dt} - \frac{4\pi^4}{15}\frac{d\beta}{dt} &= -(7!\alpha - 8!\beta)\frac{1}{\tau} \end{aligned} \qquad (9.1)$$

determining the law governing the variation of ~~change in~~ these parameters in the temperature region above 0.9°K. We seek a solution of this system of linear differential equations ~~which will be~~ in a form proportional to e^{-t/Θ_ϕ}. The quantity Θ_ϕ is the natural choice for the characteristic time determining viscosity. Multiplying by the phonon velocity c gives us the corresponding phonon mean free path λ_ϕ. Determining Θ_ϕ from the set (9.1) we obtain a quadratic characteristic equation, with roots equal to

$$\Theta_\phi \simeq 16\tau/6! \text{ and, } \tfrac{1}{3}\tau/6! \qquad (9.2)$$

In comparing the appearance of viscosity under macroscopic examination, the largest absolute magnitude of these roots must be chosen. Therefore, taking the largest values Θ_ϕ obtained for the free path of phonons, and applying (9.2) and (8.14), we obtain the following expression:

$$\frac{1}{\lambda_\phi} = \frac{45 N_c}{4\pi c^3}\left[\frac{p_0 (kT/c)^2}{h^2\rho_0}\right]^2\left[\frac{2}{15}+\frac{33}{35^2}\left(\frac{\rho_0}{\mu c}\right)^2+\frac{14A}{75}\left(\frac{\rho_0}{\mu c}\right)+A^2\right] \quad (9.3)$$

To determine the characteristic time θ_ϕ in the temperature region below 0.9°K, we obtain in analogy with (9.1) the linear differential equation

$$\frac{4\pi^4}{15}\frac{d\beta}{dt} = -\beta\left(\frac{8!}{\tau}+\frac{1}{\tau_\phi}\right) \quad (9.4)$$

In this case λ_ϕ is determined by the expression:

$$\frac{1}{\lambda_\phi} = \left(\frac{8!}{\tau}+\frac{1}{\tau_\phi}\right)\frac{15}{4\pi^4 c} = \frac{15\cdot 7!}{2\pi^3 c^2}\left[\frac{p_0 (kT/c)^2}{h^2\rho_0}\right]^2\left[\frac{2}{15}+\frac{33}{35^2}\left(\frac{\rho_0}{\mu c}\right)^2+\frac{14A}{75}\left(\frac{\rho_0}{\mu c}\right)+A^2\right]$$
$$+\frac{9\cdot 13!\,(u+2)^4}{(2\pi h)^7\cdot 2^{17}\rho_0^2 c^2}\left(\frac{kT}{c}\right)^9 \quad (9.5)$$

Substituting in the expressions (9.3) and (9.5) the numerical values of all the parameters, we finally obtain

$$1/\lambda_\phi = 3.4\cdot 10^6 T^{9/2} e^{-\Delta/kT} \quad (T > 1°K) \quad (9.6)$$
$$1/\lambda_\phi = 1.1\cdot 10^8 T^{9/2} e^{-\Delta/kT} + 5.6\cdot 10^2 T^9. \quad (T < 0.8°K)$$

Figure 1 is a graphic representation of the dependence of $1/\lambda_\phi$ on temperature*

The points along the curve of $1/\lambda_\phi$ were obtained by interpolation for temperatures between 0.8 and 1.0°K.

Let us now apply the results of the present paragraph and confirm the validity of the assumptions that we made about the velocity of ~~the~~ establishment of energ~~etic~~y equilibrium. The expression

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* In all known measurements of ~~the~~ viscosity coefficient of helium II; the size of the system has been considerably greater than the length of the phonon free path. However Andronikashvili has pointed out in experiments on heat conduction in helium II in narrow slits that the mean free path at temperature 1.3°K is already equal to the dimensions of the slits. This circumstance considerably complicates the appearance of heat conduction in helium II in narrow apertures.

(7,15) determines the time of establishment of energ~~etic~~y equilibrium for phonons, t_ϕ . From the assumption made, it follows that the process of establishment of energ~~etic~~y equilibrium occurs faster than the viscous processes, so that the time t_ϕ for the phonons, which plays a substantial role in viscosity, should be less than the smallest of the times θ_ϕ determining the relationship (9.2). It can easily be seen from (9.1) that the phonon component of the viscosity coefficient depends fundamentally on phonons with an energy of the order 7-8 kT. Comparing the times t_ϕ and θ_ϕ for phonons possessing such energy, we can confirm that at all temperatures below 1.3-1.4°K the assumption made is fulfilled.

In examining the question of the velocity of establishment of energ~~etic~~y equilibrium ~~it~~ we must ~~be~~ keep in mind that the basic parameters, and above all the scattering parameter γ determining magnitude t_ϕ , are known only extremely approximately. For this reason the most convincing verification of the assumption made can only be by experiment. The agreement between ~~the~~ computed and measured values for the viscosity coefficient obtained through the comparisons carried out below convinces us of the correctness of the assumption made. We are inclined to attribute the mutual approach of the estimated times t_ϕ and θ_ϕ at temperatures above 1.4°K to the approximate nature of the calculations made. This is indirectly confirmed by an analysis of the data on sound absorption in helium II. It should be kept in mind that at temperatures above 1.4°K, the role of the phonon component of the viscosity coefficient becomes relatively slight in comparison to the roton component, and for this reason this question is of no consequence for the temperature region above 1.4°K.

10. Temperature dependence of the phonon component of the viscosity coefficient.

Let us now turn to the calculation of the phonon component of the viscosity coefficient. First we examine the temperature region above 0.9°K. Let us solve the system (8.28) with respect to α and β, and substitute the values obtained in expression (8.5) for the deviation δn of the distribution function from its equilibrium value. Then we obtain

$$\delta n = -n_0(n_0+1)\cos\theta\cdot\sin\theta\cdot\cos\phi\,\frac{7}{6!}\left(31.2-3.44\frac{\rho c}{kT}\right)\frac{\partial u}{\partial x} \qquad (10.1)$$

In doing this we have replaced the function $P_2(\cos\theta)$ by $P_{21}(\cos\theta)\cos\phi$. We proceed in a similar way for the temperature region below 0.9°K. From equation (8.19) we find the magnitude of the parameter β, and further, using (8.7) and (8.10), we find δn:

$$\delta n = -n_0(n_0+1)\cos\theta\sin\theta\cos\phi\cdot\frac{4\pi^4\tau\tau_\phi}{15(8!\,\tau_\phi+\tau)}\,\frac{\rho c}{kT}\,\frac{\partial u}{\partial x} \qquad (10.2)$$

The z-component of momentum, differing from zero in our case and directed along x, equals

$$\sigma_{xz} = -(2\pi h)^{-3}\int \rho c\cdot\delta n\cdot\cos\theta\cdot\sin\theta\cdot\cos\phi\cdot\rho^2 d\rho\, dO. \qquad (10.3)$$

Calculating in turn the value σ_{xz} with δn in the forms (10.1) and (10.2), and comparing the result with the expression for the appropriate components of the pressure tensor $\sigma_{xz} = \eta_\phi \partial u/\partial x$, we find after simple reductions the phonon component of the ~~viscosity~~ coefficient of viscosity:

~~coefficient:~~

$$\eta_\phi = \frac{0.157(\hbar^2 p_0 c)^2 e^{\Delta/kT}}{(2\pi\mu kT)^{\frac{1}{2}} p_0^4}\left[\frac{2}{15} + \frac{33}{35}\left(\frac{p_0}{\mu c}\right)^2 + \frac{14A}{75}\left(\frac{p_0}{\mu c}\right) + A^2\right]^{-1} \quad (T > 1^\circ K)$$

$$\eta_\phi = \left(\frac{4\pi}{15}\right)^3(\hbar^2 \rho_0 c)^2\left\{8!\,(6\pi\mu kT)^{\frac{1}{2}} p_0^4\, e^{-\Delta/kT}\left[\frac{2}{15} + \frac{33}{35}\left(\frac{p_0}{\mu c}\right)^2 + \frac{14A}{75}\left(\frac{p_0}{\mu c}\right) + A^2\right] + \frac{3\;13!\,(u+2)^4}{5.\,2^{19}}\cdot\left(\frac{kT}{c}\right)^5\right\}^{-1} \quad (T < 0.8^\circ K) \qquad (10.4)$$

Substituting the numerical values of all the parameters in (10.4), we obtain as the final result

$$\eta_\phi = 8.7\;10^{-9}T^{-\frac{1}{2}}e^{+\Delta/kT} \qquad (T>1^\circ K)$$
$$\eta_\phi = 7.8\cdot10^{-10}\left(T^{\frac{1}{2}}e^{-\Delta/kT} + 4.8\cdot10^{-6}T^5\right)^{-1} \quad (T<0.8^\circ K) \qquad (10.5)$$

Thus the phonon component of the viscosity coefficient of [illegible] increases as temperature falls. At temperatures above 0.9°K, it increases in accordance with the law $e^{\Delta/kT}T^{-\frac{1}{2}}$. For temperatures below 0.7°K, where only the effect of scattering of phonons by phonons is significant, this law is replaced by the law T^{-5}. A more complex law, determined by formula (10.5) governs the intermediate area.

11. Temperature dependence of the viscosity coefficient of helium II.

In Section 6 we calculated the roton component of the viscosity coefficient η_r, which turned out to be independent of temperature. In Section 10 we set forth formulae for the phonon component of the viscosity coefficient, disclosing a sharp rise of η_ϕ with temperature drop. The actual viscosity coefficient of of helium II is equal to the sum of the roton and phonon components.

Since the coefficient for helium is the sum of a part that is constant and one that increases with fall in temperature, the viscosity coefficient ought to rise sharply with fall in temperature. No mention has previously been made of this interesting fact.

According to (6.6) and (10.4) the actual coefficient of viscosity of helium II is given as follows :

$$\eta = \frac{p_0^2 \tau_r}{15\mu} + \frac{0.157(\hbar^2 \rho_0 c)^2 e^{-\Delta/kT}}{(2\pi\mu kT)^{\frac{1}{2}} p_0^4}\left[\frac{2}{15} + \frac{33}{35^2}\left(\frac{p_0}{\mu c}\right)^2 + \frac{14A}{75}\left(\frac{p_0}{\mu c}\right) + A^2\right]^{-1} \quad (T > 1^\circ K)$$

(11.1)

$$\eta = \frac{p_0^2 \tau_r}{15\mu} + \left(\frac{4\pi^3}{15}\right)^3 (\hbar\rho_0 c)^2 \Big\{ 8!(2\pi\mu kT)^{\frac{1}{2}} p_0^4 e^{-\Delta/kT}\left[\frac{2}{15} + \frac{33}{35^2}\left(\frac{p_0}{\mu c}\right)^2 + \frac{14A}{75}\left(\frac{p_0}{\mu c}\right) + A^2\right] + \frac{3\ 13!(u+2)^4}{5.\ 2^{19}}\left(\frac{kT}{c}\right)^5 \Big\}^{-1} \quad (T < 0.8^\circ K)$$

Since the roton component of the viscosity coefficient cannot be accurately calculated, we determine this quantity by experiment. For this purpose we employ Andronikashvili's(1) data on measurements of the viscosity coefficient of ν down to a temperature of 1.35°K. Subtracting the phonon component calculated from formula (10.5) from the experimental values for the viscosity coefficient, we obtain for the temperature interval between 1.35° and 1.9°K an almost constant magnitude of 1.10^{-5} viscosity units of measurement. This we identify with the roton component of the viscosity coefficient of r. Thus we have

$$\eta_r = p_0^2 \tau_r / 15\mu = 1.\,10^{-5} \text{ ~~pou.~~ poise} \qquad (11.2)$$

The values for the constants τ_r and V_0 may be obtained from the resulting value of η_r . These constants are roughly equal to

$$\tau_r \simeq 1.5 \cdot 10^{10} \text{ sec} \cdot \text{cm}^{-3} \quad ; \qquad V_0 \simeq 0.5 \cdot 10^{-38} \text{ erg} \cdot \text{cm}^3$$

Using now the numerical values of η_r and the expression (10.5) determining the numerical value of η_{ϕ}, we write for the final expression of the viscosity coefficient

$$\eta \cdot 10^5 = 1 + 8.7 \cdot 10^{-4} T^{-\frac{1}{2}} e^{\Delta/kT} \text{ poise } (T > 1^\circ K) \qquad (11.3)$$

$$\eta \cdot 10^5 = 7.8 \cdot 10^{-5} \left(T^{\frac{1}{2}} e^{-\Delta/kT} + 4.8 \cdot 10^{-6} T^5\right)^{-1} (T < 0.8^\circ K)$$

The formulae of (11.3) determine the values of η for both temperature regions - i.e. above and below 0.9°K.

Neither one of these two formulae is valid in the neighbourhood of 0.9°K. This is explained by the fact that, at this temperature, the times characterizing the processes of emission and absorption of phonons become equal to the times characterizing the viscous processes of scattering. It might have been possible to take this circumstance into account in equation (8.3). However, since this effect is found to be significant only in a small interval of temperature of the order of 0.2°K in the neighbourhood of 0.9°K, we will not give it special consideration.

For purposes of calculation, we use the first expressions of (11.3) in the temperature regions above 1°K, and the second for temperatures below 0.8°K. We find the ~~values for the viscosity~~ coefficient of viscosity in the intermediate area by interpolation between these calculated values.

In Figure 1 a comparison is made between the values of the viscosity coeffecient obtained from (11.3) and those measured experimentally by Andronikashvili(1). The experimental values for the viscosity coefficient disclose some rise with temperature in a small region near the lambda-point. However, it should be remembered that the results that we obtained must not be extended to the temperature region in the neighbourhood of the lambda-point. In all our discussions it has been assumed that the phonon and roton gases are practically ideal. Since this assumption is not valid in the region near the lambda-point, our results are naturally not applicable in this region.

In conclusion, it must be noted that all the results obtained in the present work are of extremely limited accuracy owing to the extraordinarily crude state of our knowledge of such theoretical parameters as $\partial c/\partial \rho$, $\partial^2 c/\partial \rho^2$, $\partial^2 \Delta/\partial \rho^2$ and $\partial P_0/\partial \rho$. Of course this remark does not apply to the values obtained for the temperature dependence, which has been determined sufficiently accurately.

In his earlier works, Tisza obtained the law $\eta \approx T^5$ for the viscosity coefficient from entirely incorrect considerations. In his last work Tisza$^{(2)}$ rejected this result and obtained another temperature variation $\eta \approx T^{\frac{1}{2}}$ using an arbitrary system of postulates.

Turning to Andronikashvili's$^{(1)}$ latest experiment, and in the same connection to the results of the present work, one may be satisfied that in reality there is nothing resembling the law $\eta \approx T^{\frac{1}{2}}$ for helium II. In spite of Tisza's assertions to the contrary, helium II possesses a viscosity coefficient η with a derivative $d\eta/dT$ negative for all temperatures except for a small region near the lambda-point. In this respect it does not differ from ordinary fluids.

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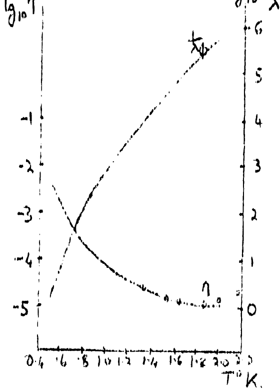
References:

(1) E.Andronikashvili. ~~J.E.T.P.,USSR~~ ZhETF, 18,429,1948.
(2) L.Tisza, Phys.Rev., 72, 839, 1947.
(For Part I of this paper, see ~~J.E.T.P.,USSR~~ ZhETF, 19,637,1949.)

Figure 1.



(o are Andronikashvili's values of η.)