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DIFFERENTIAL EQUATIONS FOR TURBULENT MOTION OF A COMPRESSIBLE FLUID

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(SUBMITTED ~~LECTURE GIVEN~~ BY A. FRIEDMANN)

Problem.

Osborne Reynolds was successful in converting the hydrodynamic equations for the motion of a viscous homogenous incompressible fluid, so that only certain averaged ~~equalized~~ values of the velocity components occur in the resulting equations. There also occur 6 quantities which characterize the condition of the turbulence at a given place and at a given time. These quantities represent 6 new unknown functions of the coordinates and the time. Reynolds' system of equations is not sufficient to determine these unknown quantities from their initial values.

The problem considered in this publication is the completion of the characterizing quantity-system of turbulence so that the knowledge of these characteristics is sufficent for the initial moment to find the corresponding ~~proper~~ values of the same functions for every additional time-point. This, we wish to do by an extension of Reynolds' ideas. The knowledge of characteristics mentioned above is in connection with the averaged ~~equalized~~ values of the kinetic and dynamic elements for the initial moment. For doing this we want to use the turbulence in the atmosphere which is thought to be the perfect compressible fluid.

Basis of the Reynolds Method.

The basis of the Reynolds Method is the comparison of a real motion with a certain "average" motion. For the latter motion all fluctuations of short duration of the velocity components are smoothed out. We understand by fluctuation in this case not only the fluctuations of the quantities with time but also space fluctuations which must be observed in transition from a arbitrary freely chosen point in the space to a neighboring point in any direction.

The smoothing-out spoken of takes place in this fashion, : ~~that~~ in the place of an arbitrary ~~a freely~~ chosen fluctuating ~~variable~~ function $\varphi(t, x, y, z)$ is taken its ~~whose~~ mean value for a certain time interval $(t - \frac{T}{2}) t + \frac{T}{2})$ τ, according to the formula:

$$\bar{\varphi} = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \varphi \, dt \tag{1}$$

or also a space-time mean value according to the formula:

$$\varphi = \frac{1}{TXYZ} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \int_{x-\frac{X}{2}}^{x+\frac{X}{2}} \int_{y-\frac{Y}{2}}^{y+\frac{Y}{2}} \int_{z-\frac{Z}{2}}^{z+\frac{Z}{2}} \varphi(t,x,y,z) \, dt \, dx \, dy \, dz \tag{2}$$

In the following section only mean values taken according to formula (2) will be observed. The four dimensional value region of the variable t,x,y,z, over which the integral in (2) extends will be called the region G in the following section.

For taking the mean values of such a nature Reynolds set up simple rules of calculation. L. F. Richardson (Weather prediction, p. 96) said on the assumptions upon which these are based, "all these assumptions are not rigidly valid, but they show good approximations as soon as the oscillations are distributed in sufficiently large numbers and at random. They would be rigidly true if it were possible to so choose a averaging ~~equalization~~ interval so that the averaged ~~equalized~~ motion could be treated as an infinitely small quantity and at the same time would be infinitely large compared to the periods of fluctuation."

The next step is to note that this shows that within the interval or value region used for taking mean values there are averaged ~~equalized~~ quantities with sufficient approximation to be handled as constants (Postulate I). From this follow the two main statements of Reynolds' algorithm:

$$\overline{\overline{\varphi}} = \overline{\varphi} \tag{3}$$

$$\overline{\overline{\varphi} \cdot \psi} = \overline{\varphi} \cdot \overline{\psi} \tag{4}$$

Further, the following distributive law is valid:

$$\overline{\varphi + \psi} = \overline{\varphi} + \overline{\psi} \tag{5}$$

and as a special or limiting case of the same, we have the following differential formula:

$$\overline{\frac{\partial(\varphi)}{\partial s}} = \frac{\partial}{\partial s}\overline{\varphi} \tag{6}$$

where s signifies any one of the 4 basic variables t,x,y,z.

Let us call such functions for which the above postulate is valid with sufficient approximation with proper choice of smoothing out values (i.e. the quantities T,X,Y,Z in the formulas (1) or (2)) as restrictedly fluctuating function.

We assume this property for the significant functions in the dynamics of the atmosphere, i.e., for the velocity components u,v,w, for the specific volume ~~w~~ ω, and for the pressure p. omega

Correlation Moments:

Let φ and ψ be any two such restrictedly ~~variable~~ fluctuating functions. We signify as "correlation moment" of the Functions φ and ψ the mean value:

$$R(\varphi,\psi) = \overline{(\varphi-\bar{\varphi})\cdot(\psi-\bar{\psi})} = \overline{\varphi\psi} - \bar{\varphi}\cdot\bar{\psi} \qquad (7)$$

These correlation moments, constructed in pairs for the various elements, are known to be the significant parameters for the statistical distribution of the various value systems of the observed functions within the region G. Therefore R (φ, φ) is the quadratic of the "spread" of φ, while the quotient

$$\frac{R(\varphi,\psi)}{\sqrt{R(\varphi,\varphi)\cdot R(\psi,\psi)}}$$

represents the "correlation factor" of φ and ψ.

The 6 characteristics introduced by Reynolds represent the correlation moments of the 3 velocity components: R(u,u) etc.

For the symbol R (φ, ψ), the following commutative and distributive laws are valid:

$$R(\psi, \varphi) = R(\varphi, \psi) \tag{8}$$

$$R(\varphi, \psi_1 + \psi_2) = R(\varphi, \psi_1) + R(\varphi, \psi_2) \tag{9}$$

Introduction of containing Moments.

Now we come, to a major expansion of Reynolds' conception. We introduce operations which are related to the statistical connections between the simultaneous conditions of two neighboring points in space, or, more general than this, between the conditions of two different "universal points."

Let there be two functions of time and of the coordinates. We set: (a)

$$\left.\begin{aligned} \varphi_1 &= \varphi(t-\tau, x-\xi, y-\eta, z-\zeta) \\ \psi_2 &= \psi(t+\tau, x+\xi, y+\eta, z-\zeta) \end{aligned}\right\}$$

and construct the correlation moment $R(\varphi_1, \psi_2)$. In taking the mean value of this only the values of the fundamental variables t, x, y, z, vary while the increments τ, ξ, η, ζ can be regarded as constant parameters.

(b)

We now further set:

$$\left.\begin{aligned} t-\tau &= t_1, \quad x-\xi = x_1, \quad y-\eta = y_1, \quad z-\zeta = z_1 \\ t+\tau &= t_2, \quad x+\xi = x_2, \quad y+\eta = y_2, \quad z+\zeta = z_2 \end{aligned}\right\}$$

The quantity $R(\varphi_1, \psi_2)$ can accordingly be regarded either as a function of the eight arguments $t, x, y, z; \tau, \xi, \eta, \zeta$ or as a function of the arguments $t_1, x_1, y_1, z_1; t_2, x_2, y_2, z_2$. In the first case we signify the function with the symbol $R_{\varphi,\psi}$, and in the second case with $\bar{R}_{\varphi,\psi}$. Therefore the following definitions are valid: (10)

$$R(\varphi_1, \psi_2) = R_{\varphi,\psi}(\tau, \xi, \eta, \zeta; t, x, y, z) = \bar{R}_{\varphi,\psi}(t_1, x_1, y_1, z_1; t_2, x_2, y_2, z_2)$$

We call the quantities $R_{\varphi,\psi}$ "containing moments", because a "containing tendency" of the variations from the relative mean finds expression in these "moments", to be exact, this "tendency" exists with respect to time as well as space variations.

Calculation Rules for the Containing Moments.

The following properties are contained in the symbol $R_{\varphi,\psi}$: First, $R_{\varphi,\psi}$ is not commutative with respect to φ and ψ. Instead there is the relationship: (11)

$$R_{\psi,\varphi}(\tau, \xi, \eta, \zeta; t, x, y, z) = R_{\varphi,\psi}(-\tau, -\xi, -\eta, -\zeta; t, x, y, z)$$

On the other hand, the same property follows from the distributive law for $R(\varphi, \psi)$ for the newly introduced symbol $R_{\varphi,\psi}$: (12)

$$R_{f,\varphi+\psi} = R_{f,\varphi} + R_{f,\psi}$$

When the increments τ, ξ, η, ζ are reduced to zero, the containing moment $R_{\varphi,\psi}$ goes into the Reynolds correlation moment $R(\varphi, \psi)$:

$$R_{\varphi,\psi}(0,0,0,0;t,x,y,z) = R(\varphi,\psi) \qquad (13)$$

Further, the following differential formulas are valid:

$$\left.\begin{aligned} R_{\varphi,\frac{\partial\psi}{\partial s}} &= \frac{1}{2}\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial\sigma}\right) R_{\varphi,\psi} \\ R_{\frac{\partial\varphi}{\partial s},\psi} &= \frac{1}{2}\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial\sigma}\right) R_{\varphi,\psi} \end{aligned}\right\} \qquad (14)$$

where s, as before, represents any one of the four variables t,x,y,z, while σ represents the resulting increment (τ, respectively ξ, η, ζ).

<u>Proof:</u> From the equation of definition (10) and with help of (a) and (b) we obtain: (c)

$$R_{\varphi,\frac{\partial\psi}{\partial t}} = R\left(\varphi(t_1,x_1,y_1,z_1), \frac{\partial\psi(t_2,x_2,y_2,z_2)}{\partial t_2}\right) = R\left(\varphi_1, \frac{\partial\psi_2}{\partial t_2}\right)$$

On the basis of the distributive law we further have: (d)

$$R\left(\varphi_1, \frac{\partial\psi_2}{\partial t_2}\right) = \frac{\partial}{\partial t_2} R(\varphi_1,\psi_2) = \frac{\partial}{\partial t_2}\bar{R}_{\varphi,\psi}$$

Thus we get the differential formulas: (e)

$$R_{\varphi\frac{\partial\psi}{\partial s}} = \frac{\partial}{\partial s_2}\bar{R}_{\varphi,\psi}$$

and in the same manner: $R_{\frac{\partial \varphi}{\partial s},\psi} = \frac{\partial}{\partial s_1} \overline{R}_{\varphi,\psi}$

We arrive at formula (14) in that we express the derivatives of $\overline{R}_{\varphi,\psi}$ with respect to s_1 and s_2 by derivatives of $R_{\varphi,\psi}$ with respect to s and σ. This we do on the basis of the relations (10) and (b).

In conjunction with the relationships developed above, we give two additional formulas which relate certain mean values to the symbol $R_{\varphi\psi}$.

Now we write formula (13) in abbreviated form:

$$R(\varphi,\psi) = (R_{\varphi,\psi})_0 \tag{15}$$

From this we obtain, using (7)

$$\overline{\varphi\psi} = \overline{\varphi}\cdot\overline{\psi} + (R_{\varphi\psi})_0$$

From (15) we further get with help of the differential formulas (6) and (14):

$$\overline{\varphi\frac{\partial\psi}{\partial s}} = \overline{\varphi}\cdot\frac{\partial\overline{\psi}}{\partial s} + \frac{1}{2}\frac{\partial}{\partial s}(R_{\varphi,\psi})_0 + \frac{1}{2}\left(\frac{\partial}{\partial\sigma}R_{\varphi\psi}\right)_0 \tag{16}$$

Supplementary Assumptions

The material shown above requires no further **hypotheses beyond** the assumptions that are the basis of Reynolds' theory. Now we must make two further limitations. Without these limitations our algorithm for productive treatment of the hydrodynamic equations would not be sufficient. This can be seen in that we cannot operate with expressions of the form using calculation rules we have set up thus far.

First, we must postulate that the containing moment can only assume values noticeably different from zero in the immediate neighborhood of the ~~global~~ world point (t, s, y, z) - let us say in a small value region Γ of the increments τ, ξ, η, ζ.

It must be so, that within the value-region Γ all smoothed-out functions ($\bar{\varphi}$ etc.) can be handled as constants. (This is Postulate II and makes a demand similar to the one made in Postulate I for the region G)

Secondly, we assume that apart from the moments of the order $R_{\varphi,\psi}$ and their derivatives we can overlook moments of higher order. (This is postulate III.)

Such is the case as soon as we admit that the "Smoothing-out values", which are formed by the process of taking the mean of smoothed-out oscillations $(\varphi' = \varphi - \bar{\varphi})$ can be observed as a system of waves not only of short-periods (as is demanded by postulates I and II) but also of small amplitude.

On the basis of the last postulate we can set up approximation formulas for the calculation of mean values and of correlation moments for combined expressions which depend on several variable functions.

Without going into these general formulas further, we show a very special expression which we need for our immediate purposes:

$$R(\varphi_1, \varphi_2, \varphi_3) = \bar{\varphi}_2 \cdot R(\varphi_1, \varphi_3) + \bar{\varphi}_3 \cdot R(\varphi_1, \varphi_2) \quad (17)$$

Proof: Let us introduce the correlation moments of the third degree in which we set:

$$R(\varphi_1, \varphi_2, \varphi_3) = \overline{(\varphi_1 - \bar{\varphi}_1) \cdot (\varphi_2 - \bar{\varphi}_2) \cdot (\varphi_3 - \bar{\varphi}_3)}$$

and thus with the help of the Reynolds' calculation rules we can easily verify the following exact formula:

$$R(\varphi_1, \varphi_2, \varphi_3) = \overline{\varphi}_2 \cdot R(\varphi_1, \varphi_3) + \overline{\varphi}_3 \cdot R(\varphi_1, \varphi_2) + R(\varphi_1, \varphi_2\, \varphi_3)$$

If we neglect the third term on the right on the basis of the above postulate, then we have the approximate formula (17).

Formula (17) gives us the transformation we have been seeking for the expression $R_{\varphi, f \frac{\partial \psi}{\partial s}}$ on the basis of the definition of the containing moment according to Formula (10) with the help of Postulate II and the differential formula (14):

$$R_{\varphi, f \frac{\partial \psi}{\partial s}} = \frac{\partial \overline{\psi}}{\partial s} \cdot R_{\varphi, f} + \frac{1}{2} \overline{f} \cdot \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial \sigma} \right) R_{\varphi, \psi}$$

(18)

$$R_{f \frac{\partial \psi}{\partial s}, \varphi} = \frac{\partial \overline{\psi}}{\partial s} \cdot R_{f, \varphi} + \frac{1}{2} \overline{f} \cdot \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial \sigma} \right) R_{\psi, \varphi}$$

These formulas are obviously a generalization of the formulas (14) which we receive again if we set f = 1 in (18).

<u>Setting Up the Differential Equations of Turbulent Motion</u>

The operations mentioned above will now be applied to thehydrodynamic equations.

Let us at first limit ourselves to the case of the adiabatic motion of an ideal heavy compressible fluid. Let us write the equations of motion in the Euler form:

$$\left.\begin{aligned}
\frac{du}{dt} &= -\omega\frac{\partial p}{\partial x} \\
\frac{dv}{dt} &= -\omega\frac{\partial p}{\partial y} \\
\frac{dw}{dt} &= -\omega\frac{\partial p}{\partial z} - g \\
\frac{d\omega}{dt} &= \omega\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\
\frac{dp}{dt} &= -\varkappa p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)
\end{aligned}\right\} \qquad (19)$$

In order to make it easier to have a perspective of the following observations, we will designate the five unknown functions, u,v,w,ω,p respectively by $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ and will write the five equations (19) in the form:

$$\frac{\partial \varphi_i}{\partial t} = F_i \qquad (20)$$

or, by setting $\frac{\partial \varphi_i}{\partial t} - F_i = \Phi_i$ we can write even more briefly:

$$\Phi_i = 0 \qquad i = (1,2,3,4,5) \qquad (20a)$$

We will now try to transform these equations so, that in the resulting system the ~~variable~~ fluctuating functions φ_i no longer occur and we have, instead, the mean values $\overline{\varphi}_i$ and the characteristics of the turbulence R_{φ_i, φ_k}.

In connection with this last statement, we want to prove that there [characteristics] satisfy the initially set conditions for a complete system of characteristics.

For the transformation mentioned we have the use of two operations: (1) the simple taking of mean values, (2) the formation of containing moments.

Whenever it is not necessary to make particular indication of the functional dependence of the quantities $R_{\varphi,\psi}(v, \xi, \eta, \zeta; t, x, y, z)$ on their eight arguments, we will replace this symbol with the simple parenthetical expression (φ, ψ).

We now form the following equations:

$$\Phi_i = 0 \quad \text{(5 equations)} \tag{21}$$

$$(\varphi_i, \Phi_k) = 0 \quad \text{(25 equations)} \tag{22}$$

$$(\Phi_i, \varphi_k) = 0 \quad \text{(25 equations)} \tag{23}$$

By using successively the calculation rules mentioned above the equations (21) - (23) can obviously be transformed in the manner we wish.

The resulting equations then contain the following quantities: (1) the five mean values $\overline{\varphi_i}$ and the derivatives of these quantities with respect to time and the coordinates.

(2) the 25 containing moments (φ_i , φ_k) and their derivatives, first with respect to the time and the coordinates,

secondly with respect to the increments τ, ξ, η, ζ;

(3) the special values, which the functions mentioned in (2) and their derivatives assume for the special value system $\tau = \xi = \eta = \zeta = 0$.

The functions shown in (1) and (2) must also satisfy the following supplementary conditions:

(a) the five quantities $\overline{\varphi}_i$ depend solely on the time and the coordinates, but are independent of the increments. This gives us 5 x 4 = 20 equations of the form:

$$\frac{\partial \overline{\varphi}_i}{\partial \sigma} = 0 \qquad (24)$$

(b) (φ_i, φ_k) are made subject to the condition (11) which produces 25 finite relations among 50 functions $R_{\varphi_i, \varphi_k}(\tau, \xi, \eta, \zeta; t, x, y, z)$ and $R_{\varphi_i, \varphi_k}(-\tau, -\xi, -\eta, -\zeta; t, x, y, z)$. These relations we write in brief as follows:

$$(\varphi_k, \varphi_i) = (\varphi_i, \varphi_k) \qquad (25)$$

Conversion and Discussion of the Equations Produced

In order to make the connections between the equations (21) - (25) clear, we write the hydrodynamic basic equation (19) in the form (20). From this the equations (21) - (23) can be thus represented:

$$\frac{\partial \overline{\varphi}_i}{\partial t} = \overline{F}_i \qquad (26)$$

$$\left(\varphi_i, \frac{\partial \varphi_k}{\partial t}\right) = (\varphi_i, F_k) \qquad (22a)$$

$$\left(\frac{\partial \varphi_i}{\partial t}, \varphi_k \right) = \left(F_i, \varphi_k \right) \qquad (23a)$$

We develop the two parenthetic expressions on the left according to the differentiation formulas (14) and then, after addition and subtraction, we get:

$$\frac{\partial}{\partial t}(\varphi_i, \varphi_k) = (\varphi_i, F_k) + (F_i, \varphi_k) \qquad (27)$$

$$\frac{\partial}{\partial \tau}(\varphi_i, \varphi_k) = (\varphi_i, F_k) - (F_i, \varphi_k) \qquad (28)$$

Since the expressions F_i etc. do not contain derivatives with respect to t, then - on the basis of the generalized differentiation formulas (18) - the expressions on the right in (27) and (28) cannot contain derivatives with respect to t or τ but only those with respect to $x, y, z, \xi, \eta, \zeta$. The system (27) then represents the result of the elimination of all differential quotients [sic!] with respect to τ from the equations (22) - (23). The system (28) is a result of the elimination of the differential quotients with respect to t.

From this we may conclude the initial values of the containing moments (φ_i, φ_k) produced by time point t = 0 can most definitely not be observed as arbitrary functions of x, y, $z, \tau, \xi, \eta, \zeta$, since these values must satisfy the equations (28) under any condition.

The systems of equations (26), (27), (28) is equivalent to the system (21) - (23). Therefore in order to determine the functions sought we can hold on to the equations (26) - (28) and the secondary conditions (24) - (25).

In connection with the latter it can be shown that it is sufficient to introduce these conditions as the conditions to be imposed on the <u>initial values of the unknown functions!</u>

Let there be given the values of $\overline{\varphi}_i$ for the initial moment t = 0, or generally speaking, for any one time point and these values shall be functions of x,y,z. Let there be given the (φ_1, φ_2) for the same time point, for a definite value of τ e. g. $(\tau = 0)$ as functions of x, y, z, ξ, η, ζ. Accordingly the conditions (24) for the initial values are satisfied. Further, they must also satisfy the conditions (25).

With these assumptions the system of the 25 equations (28) represents a normal system for which the Cauchy Problem can be solved, i.e. the 25 unknown functions (φ_i, φ_k) can be ascertained on the basis of these equations as functions of the 7 arguments x,y,z, ξ, η, ζ and τ.

This ascertainment we will call <u>Operation A.</u>

On the other hand the 30 unknown quantities $(\varphi_i, \varphi_k), \overline{\varphi}_i$ coming from the same initial values through the 25 equations (27) with the aid of the 5 equations (26), can be represented as functions of the arguments x,y,z, ξ, η, ζ and t. This we will refer to as <u>Operation B</u>. (The system of the 30 equations (26) - (27) differs from a normal system of differential equations in the ordinary sense in that in the right-hand expressions the values of the unknown functions and their derivatives with respect to

ξ, η, ζ occur for the general value system ξ, η, ζ as well as for the special value system $\xi = \eta = \zeta = 0$. This condition does not make any difference, however, for the setting up and solution of this problem analogous to the Cauchy Problem.)

On the basis of the proceeding observation we can now make the statement: if a value system exists at all for the desired 30 functions of the 8 arguments $t, x, y, z, \tau, \xi, \eta, \zeta$ which satisfies the equations (24) - (28) and which takes on the initial values demanded for $t = 0$, $\tau = 0$, then this value system can clearly be found through successive application of Operations A and B. It makes no difference in what order the operations are applied. (It is not desired to assert that the initial values of the 25 containing moments actually can be treated as arbitrary functions independent on one another. The opposite statement can be proved in the following manner. If a system of these quantities exists as a function of the 8 arguments s and σ, which satisfy the equations (27) and (28), then the following 25 equations must exist:

$$\frac{\partial}{\partial \tau}\left[(\varphi_i, F_k) + (F_i, \varphi_k)\right] = \frac{\partial}{\partial \tau}\left[(\varphi_i, F_k) - (F_i, \varphi_k)\right]$$

If these expressions are developed and all the differential quotients with respect to t or τ are replaced by their expressions from the equations (26) - (28) then we get a system of equations of the second order which only contains derivatives with respect to the 6 arguments $x, y, z, \xi, \eta, \zeta$. Each equation of this type produces, however, one of the conditions which were placed on the initial values (valid for $t = \tau = 0$).

Only when the conditions so constructed are actually satisfied, does the use of the Operations AB and BA produce one and the same result for the whole value region of the 8 arguments).

If we are not further interested in the determination of the characteristics of turbulence and are only interested to ascertain the quantities $\overline{\varphi}$, then we can forget Operation A. From this standpoint we can be content with the determination of the Containing moments for the special value τ=0 (in the case of the unlimited variables ξ, η, ζ). Thus we get a complete system of characteristics which, with the exception of the coordinates and time, depends only on the three further arguments: ξ, η, ζ. We should like to remark that in view of the relation (25) the system is reduced to 15 basically different functions.

We must also prove that, as soon as the conditions (24) and (25) for the initial values t=τ=0 are satisfied, the conditions also have validity for the region enlarged by the operations A, respectively B.

We need only say that for the equation (24) the $\overline{\varphi_i}$ remain unchanged by the operation A and that, on the other hand, the expressions F_i which are on the right side of the equation (26) are independent of the arguments σ. From this, it can be seen that our assertion in connection with operation B is correct.

To prove the resulting assertion for the relation (25) it is enough to show -- as far as these relations together with the equations (24) are valid for a region limited by determination of a definite pair of values t, τ for this region there are the following equations:

$$\left.\begin{aligned} \frac{\partial}{\partial t}\left[(\varphi_i, \varphi_k) - (\varphi_k, \varphi_i)_-\right] = 0 \\ \frac{\partial}{\partial \tau}\left[(\varphi_i, \varphi_k) - (\varphi_k, \varphi_i)_-\right] = 0 \end{aligned}\right\} \qquad \text{(f)}$$

Through these equations an extension of the relation (25) takes place beyond the region in question.

We arrive at these equations in the following manner: First, the differentiation of the equations (25) results in:

$$\left.\begin{aligned} \left(\frac{\partial}{\partial s}(\varphi, \psi)\right)_- = \frac{\partial}{\partial s}(\varphi, \psi)_- = \frac{\partial}{\partial s}(\psi, \varphi) \\ \left(\frac{\partial}{\partial \sigma}(\varphi, \psi)\right)_- = -\frac{\partial}{\partial \sigma}(\varphi, \psi)_- = -\frac{\partial}{\partial \sigma}(\psi, \varphi) \end{aligned}\right\} \qquad \text{(g)}$$

In this case s means one of the three quantities x,y,z and σ means one of the three quantities

With the help of these equations we have from the formulas (18):

$$\left.\begin{aligned} \left(\varphi, f\frac{\partial \psi}{\partial s}\right)_- = \left(f\frac{\partial \psi}{\partial s}, \varphi\right) \\ \left(f\frac{\partial \psi}{\partial s}, \varphi\right)_- = \left(\varphi, f\frac{\partial \psi}{\partial s}\right) \end{aligned}\right\} \qquad \text{(h)}$$

By taking into account the construction of the functions we get:

$$\left.\begin{aligned}(\varphi_i, F_k)_- &= (F_k, \varphi_i)\\ (F_i, \varphi_k)_- &= (\varphi_k, F_i)\end{aligned}\right\} \qquad \text{(i)}$$

With the help of this formula we get from the equations (27) and (28) the relations (f) given above.

Simplification of the Preceding Method.

The formulas given above give a complete system of the equations for the definition of the characteristics of turbulence and of the average motion. They are still very complicated, on the one hand because there are a great number of unknown functions and on the other because there is a double number of independent variables.

We can reach ~~a~~ certain simplification of the notations above by introducing of the containing moments for infinitely near points of the four-dimensional continunm, and we set:

$$R_S(\varphi, \psi) = \left(\frac{\partial R_{\varphi,\psi}}{\partial \sigma}\right)_0 \qquad (29)$$

With the help of the functions developed above, we have:

$$R_s(\psi, \varphi) = -R_s(\varphi, \psi)$$

$$R_s(\varphi, \varphi) = 0$$

$$R_s(\varphi, \psi_1 + \psi_2) = R_s(\varphi, \psi_1) + R_s(\varphi, \psi_2)$$

$$\overline{\varphi \frac{\partial \psi}{\partial s}} = \bar{\varphi} \frac{\partial \bar{\psi}}{\partial s} + \frac{1}{2} \frac{\partial R(\varphi, \psi)}{\partial s} + \frac{1}{2} R_s(\varphi, \psi) \tag{30}$$

$$R\left(\varphi, \frac{\partial \psi}{\partial s}\right) = \frac{1}{2} \frac{\partial R(\varphi, \psi)}{\partial s} + \frac{1}{2} R_s(\varphi, \psi)$$

$$R\left(\varphi, f \frac{\partial \psi}{\partial s}\right) = \frac{\partial \bar{\psi}}{\partial s} R(\varphi, f) + \frac{1}{2} \bar{f} \frac{\partial R(\varphi, \psi)}{\partial s} + $$

$$+ \frac{1}{2} \bar{f} \cdot R_s(\varphi, \psi)$$

If for simplicity's sake we take the linear adiabatic motion of an ideal compressible fluid in the absence of ~~out-side~~ external forces, then we have the following system of equations:

$$\frac{\partial u}{\partial t} = -\omega \frac{\partial p}{\partial x} - u \frac{\partial u}{\partial x}$$

$$\frac{\partial \omega}{\partial t} = \omega \frac{\partial u}{\partial x} - u \frac{\partial \omega}{\partial x} \tag{31}$$

$$\frac{\partial p}{\partial t} = -\kappa p \frac{\partial u}{\partial x} - u \frac{\partial p}{\partial x}$$

$\{ \kappa$: kappa

Now we take the mean value of these three equations and form the correlation moments.

In the resulting equations there generally appear the characteristics R_t and R_x.

If we know only the initial conditions of the turbulent motion then we cannot determine the R_x but we can determine the R_t. For this reason it is desirable to transform the equations (31) so that the results contain no R_t. In view of this remark we obtain from the system (31), with the help of the formulas (30), a system (32) of 9 equations for the differential quotients $\frac{\delta u}{\delta t}$, $\frac{\delta \omega}{\delta t}$, $\frac{\delta p}{\delta t}$, $\frac{\delta R(u,u)}{\delta t}$, $\frac{\delta R(p,u)}{\delta t}$ $\frac{\delta R(\omega,u)}{\delta t}$, $\frac{\delta R(\omega,\omega)}{\delta t}$, $\frac{\delta R(\omega,p)}{\delta t}$, $\frac{\delta R(p,p)}{\delta t}$ (whereby we write, for brevity $\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}$), where the quotients contain, besides the quantities $\bar{u}$, $\bar{\omega}$ ....R(p,p) s the tree further quantities $R_x(\omega,u)$, $R_x(p,u)$, $R_x(\omega,p)$,

We have then 9 equations for 12 unknowns and we therefore do not get a complete system in this way. The number of indefinite functions remaining has been reduced considerable in contrast to the results of the simple mean-taking - that is, the results of Reynolds' approach.

We can achieve a complete normal system from the suggested system of equations (32) if we arbitrarily control the 3 functions $R_x(\omega, u)$, $R_x(p, u)$ $R_x(\omega, p)$. In this manner we have obtained by integrating, as a particular solution the following case of turbulent motion:

$$\bar{u} = b;\ p = \frac{b^2}{2} + \frac{\alpha t^2 + \beta t + \gamma}{2} + c;\ \bar{\omega} = 1$$

$$R(u, u) = \alpha t^2 + \beta t + \gamma;\quad R(\omega, \omega) = \psi(x - bt)$$

$$\begin{aligned} R(p, p) = \alpha x^2 - (2\alpha\beta t + 2a - b\beta)x + \\ + \alpha b^2 t^2 + (2ab - b^2\beta)t + C \end{aligned} \qquad (33)$$

$$R(p, u) = -(\alpha t + \frac{\beta}{2})x + \alpha b t^2 + a t;$$
$$R_x(p, u) = \alpha t + \frac{\beta}{2}$$

$$R(\omega, p) = R(\omega, u) = R_x(\omega, p) = R_x(\omega, u) = 0$$

C, a, b, c, α, β, γ are arbitrary constants in this case and ψ is an arbitrary function of their arguments.

The fluid behaves, in a given case, like an incompressible fluid of the density 1, in mean motion. The mean motion represents, under these circumstances, a uniform translation of the fluid as a rigid body.

As we have noted, the equations valid for the characteristic R_s are not a complete system. The missing relationship between the characteristics of the turbulence must be determined experimentally.

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