EVALUATION OF THE INTENSITY OF A WAVE DIFFRACTED FROM A DIELECTRIC SPHERE

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Nonhomogeneities in the atmosphere in the form of cloud formations (globules) affect the propagation of ultrashort waves. In connection with this, we consider the phenomenom of diffraction in the simplest case from a sphere with a dielectric constant $\varepsilon^{(i)} = 1 + \delta\varepsilon$, which differs little ($\delta\varepsilon \ll 1$) from the dielectric constant of the surrounding medium (for which $\varepsilon^{(a)} = 1$).

Let a plane electromagnetic wave, propagating on the negative side of the z-axis with vibrations of the electric field along the x-axis (unit amplitude of vibrations is assumed), be incident upon a dielectric sphere of radius R with its center O at the origin of the coordinate system.

In the spherical system (r, ϕ, θ), of coordinates (with origin at O), the components of the electric field of the diffracted wave are[1, 2]

$$E_\theta = \frac{1}{k^{(a)}r} \cos\phi \, e^{i\omega t} \left\{ \sum_{l=1}^{\infty} A_l^{(1)} \zeta_l(k^{(a)}r) \frac{d}{d\theta} P_l(\cos\theta) + \right. \left. + i \sum_{l=1}^{\infty} A_l^{(2)} \zeta_l(k^{(a)}r) \frac{P_l^{(1)}(\cos\theta)}{\sin\theta} \right\}. \quad (1)$$

Here $k^{(a)} = \frac{2\pi}{\lambda_0} = \frac{\omega}{c}$ is the wave number of the external medium (with respect to the sphere). Inside the sphere

$$k^{(i)} = \frac{\omega}{c} n^{(i)}, \quad n^{(i)} = \sqrt{\varepsilon^{(i)}} = 1 + \frac{1}{2}\delta\varepsilon .$$

The cylindrical functions

$$\psi_l(x) = \sqrt{\frac{\pi x}{2}}\, J_{l+\frac{1}{2}}(x), \quad \zeta_l(x) = \sqrt{\frac{\pi x}{2}}\, H^{(2)}_{l+\frac{1}{2}}(x)$$

are Debye normed.

We limit ourselves to consideration of the field on the z-axis only, for which

$$\theta = 0, \quad \frac{d}{d\theta} P_l^{(1)}(\cos\theta) = \frac{l(l+1)}{2}, \quad E_r = 0;$$

E_ϕ differs from E_θ only in that $\cos\phi$ is replaced by $\sin\phi$ in expression (1).

In this work, we are interested in the electromagnetic field only at a considerable distance from the ~~field~~ sphere ($r \gg R$). It is known that for

$$r \rightarrow \infty \quad \zeta_\ell(k^{(a)}r) = j^{\ell+1} e^{-jk^{(a)}r}, \quad \zeta_\ell'(k^{(a)}r) = j^{\ell} e^{-jk^{(a)}r}$$

Formula (1) takes on the form

$$E_\Theta = \frac{1}{k^{(a)}r} \cos\phi \, e^{j(\omega t - k^{(a)}r)} \sum_{\ell=1}^{\infty} \frac{\ell(\ell+1)}{2} j^{\ell} \left(A_\ell^{(1)} - A_\ell^{(2)}\right), \quad (2)$$

and moreover

$$A_\ell^{(1)} = -j^{\ell-1} \frac{2\ell+1}{\ell(\ell+1)} \frac{\psi_\ell(k^{(a)}R)\,\psi_\ell'(k^{(i)}R) - n^{(i)}\psi_\ell(k^{(i)}R)\,\psi_\ell'(k^{(a)}R)}{\zeta_\ell(k^{(a)}R)\,\psi_\ell'(k^{(i)}R) - n^{(i)}\psi_\ell(k^{(i)}R)\,\zeta_\ell'(k^{(a)}R)} \quad (3)$$

The coeffieient $A_\ell^{(2)}$ is obtained from this expression if the constant $n^{(i)}$ in the numerator and denominator of the right-hand side are shifted from the subtrahend to the minuend.

A further simplification is obtained if we assumed that for $\delta\varepsilon \ll 1 : (k^{(i)} - k^{(a)})R \gg 1$ and by using the identity $\psi(x)\zeta'(x) - \zeta(x)\psi'(x) = \frac{1}{j}$.

Disregarding infinitesimals, beginning with those of the second order (with respect to $\delta\varepsilon$), we find after all calculations that the difference

$$A_\ell^{(1)} - A_\ell^{(2)} = -j^{\ell} \frac{2\ell+1}{\ell(\ell+1)} \psi_\ell(x_0)\,\psi_\ell'(x_0)\,\delta\varepsilon, \quad (4)$$

where x_0 has been set equal to $k^{(a)}R$ for brevity.

It is well known that the series (1) and (2) converge very slowly if the wavelength of the incident wave λ_0 is considerably less than the radius of the sphere R. This is what obtains in the problem of diffraction of ultrashort waves discussed here.

By assumption kR $\gg$ 1 and therefore for cylindrical functions in the equality (4), we can use Debye's asymptotic formulas. The summing with respect to the index ℓ in expression (2) is divided into three parts from 1 to ℓ_1, from $\ell_1 + 1$ to ℓ_2 and from $\ell_2 + 1$ to ∞, because of the different form of the Debye formulas for the three regions:

$$l + \frac{1}{2} < x_0, \quad l + \frac{1}{2} \approx x_0, \quad l + \frac{1}{2} > x_0.$$

Comparing the absolute value of the ratio of two consecutive members of the asymptotic series in the first region

$$\mu = \frac{5}{24}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\eta, \quad \left[\eta = \frac{2\cot\tau_0}{x_0 \sin\tau_0}, \quad \cos\tau_0 = \frac{l+\frac{1}{2}}{x_0}\right]$$

with a similar ratio for the second region

$$V = 6^{\frac{1}{3}}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\delta \left[\delta = \frac{x_0 - l - \frac{1}{2}}{x_0^{\frac{1}{3}}}\right].$$

and setting approximately at the boundary of these regions

$$\delta_1 = 2^{-\frac{1}{3}}\eta_1^{-\frac{2}{3}}$$

we find

$$\eta_1 = 3^{11/10}\left(\frac{8}{5}\right)^{3/5}\frac{1}{\pi^{3/5}}\left[\Gamma\left(\frac{2}{3}\right)\right]^{6/5}. \qquad (5)$$

By repeating the same procedure for the third ($l + \frac{1}{2} > x_0$) and second regions, we find the desired indices l_1 and l_2, which determine the boundaries of the three regions:

$$l_1 = x_0 - \frac{1}{2} - 2^{-\frac{1}{3}}\eta_1^{-\frac{2}{3}}x_0^{\frac{1}{3}}, \quad l_2 = x_0 - \frac{1}{2} + 2^{-\frac{1}{3}}\eta_1^{-\frac{2}{3}}x_0^{\frac{1}{3}}. \qquad (6)$$

In accordance with the above, the component E_Θ (see equation 2) consists of three sums: S_i ($i = 1, 2, 3$). We will calculate the first sum, using the following Debye formulas ($l + \frac{1}{2} < x_0$): $\psi_l(x_0) = \frac{\cos\left(x_0 f_0 - \frac{\pi}{4}\right)}{\sqrt{\sin\tau_0}}$ and

$\psi_l'(x_0) = \cos\left(x_0 f_0 + \frac{\pi}{4}\right)\sqrt{\sin\tau_0}$,

where $\cos\tau_0 = \frac{l+\frac{1}{2}}{x_0}$ and $f_0 = \sin\tau_0 - \tau_0\cos\tau_0$.

Then, taking formula (b) into consideration, we have as a first approximation

$$S_1 = -\frac{1}{4}\frac{R}{r}\delta\varepsilon\cos\phi\, e^{j(\omega t - k^{(a)}r)}\sum_{l=1}^{l_1}(-1)^l\cos\tau_0\left[e^{2jx_0 f_0} + e^{-2jx_0 f_0}\right]. \qquad (7)$$

Let us evaluate the magnitude of the sum in the ~~ride~~ right side of this equation. Since by assumption x_o is much greater than 1, $\mathcal{J}_o$ changes by a small value ($\ll 1$) for a unit increase of ℓ. Therefore, by grouping the components of this sum in pairs, we have (for an even ℓ_1) $\sum_{\ell=1,2,3\ldots}^{\ell_1}(-1)^{\ell}\cos\mathcal{J}_o(\ell)\left[e^{2jx_of_o(\ell)}+e^{-2jx_of_o(\ell)}\right]\approx$

$$\approx\sum_{\ell=1,3,5\ldots}^{\ell_1-1}\left[e^{-2j\mathcal{J}_o(\ell)}-1\right]e^{2jx_of_o(\ell)}\cos\mathcal{J}_o(\ell)+\sum_{\ell=1,3,5\ldots}^{\ell_1-1}\left[e^{2j\mathcal{J}_o(\ell)}-1\right]e^{-2jx_of_o(\ell)}\cos\mathcal{J}_o(\ell).$$

Here, we have taken into consideration that

$$f_o(\ell+1)-f_o(\ell)=\mathcal{J}_o\sin\mathcal{J}_o\Delta\mathcal{J}_o \quad \text{and} \quad \sin\mathcal{J}_o\Delta\mathcal{J}_o=-\frac{1}{x_o}.$$

If ℓ_1 is odd, the summing in the left-hand side of the equality should be made up to $\ell=\ell_1-1$ and then the component corresponding to the index ℓ_1 should be added. In the two sums of the right-hand side of the latter equality, the index ℓ "skips" to the values 1, 3, 5, ... We can again return to the values 1, 2, 3, ..., if we use the easily-proved relationship

$$\sum_{\ell=1,3,5\ldots}^{\ell_1-1}2F[\mathcal{J}_o(\ell)]\,e^{-j\xi x_of_o(\ell)}\cos\mathcal{J}_o(\ell)\approx\sum_{\ell=1,2,3\ldots}^{\ell_1}F[\mathcal{J}_o(\ell)]\,e^{2j\xi x_of_o(\ell)},$$

where $F[\mathcal{J}_o(\ell)]$ is a slowly varying function of ℓ, and ξ assumes the values $+1$ and -1.

In the case under consideration, $F[\mathcal{J}_o(\ell)]=-j\,\xi\,\sin[\mathcal{J}_o(\ell)]$. We shall use the methods of the calculus of finite differences (3) in order to calculate the sum in the right-hand side of the last relationship.

We introduce the function

$$\Phi(\ell)=\frac{1}{2}e^{j\xi\mathcal{J}_o(\ell)}e^{2j\xi x_of_o(\ell)},$$

which has the property that $\Delta\Phi(\ell)=\Phi(\ell+1)-\Phi(\ell)$ represents approximately the function to be summed. Actually, calculations show that for a slowly varying function $\mathcal{J}_o(\ell)$

$$\Delta\Phi(\ell)\approx\frac{1}{2}e^{j\xi\mathcal{J}_o(\ell)}\left[e^{2j\xi x_of_o(\ell+1)}-e^{2j\xi x_of_o(\ell)}\right]\approx-j\xi e^{2j\xi x_of_o(\ell)}\sin[\mathcal{J}_o(\ell)]$$

and consequently

$$\sum_{\ell=1}^{\ell_1} F[\mathcal{T}_0(\ell)]\, e^{2j\xi x_0 f_0(\ell)} = \Phi(\ell+1) - \Phi(\ell).$$

This approach is used to calculate the sum

$$S_1 = -\frac{1}{4}\frac{R}{r}\,\delta\varepsilon \cos\phi\, e^{j(\omega t - k^{(a)}r)} \left\{ \cos\left[2x_0 f_0(\ell+1) + \mathcal{T}_0(\ell+1)\right] + \cos 2x_0 \right\}. \tag{8}$$

The order of the sum S_1, as this formula shows, is essentially determined by the quantity $\frac{R}{r}\delta\varepsilon$. For the transitional case, when $\ell + \frac{1}{2} \approx x_0$, we retain the first three members in the Debye formulas:

$$\psi_\ell(x_0) = \frac{1}{3\sqrt{2\pi}}\left[\alpha x_0^{\frac{1}{6}} + \beta x_0^{\frac{5}{6}} - \frac{1}{2}\beta(2\ell+1)x_0^{-\frac{1}{6}} + \ldots\right],$$

where $\alpha = 6^{\frac{1}{3}} \sin\frac{\pi}{3}\,\Gamma\left(\frac{1}{3}\right)$, $\beta = 6^{\frac{2}{3}} \sin\frac{2\pi}{3}\,\Gamma\left(\frac{2}{3}\right)$, and

$$\gamma = 6^{\frac{3}{3}} \sin\frac{3\pi}{3}\,\Gamma\left(\frac{3}{3}\right) = 0.$$

The well-known formula for cylindrical functions

$$Z_p' = \frac{1}{2} Z_{p-1} - \frac{1}{2} Z_{p+1}$$

yields

$$\psi_\ell'(x_0) = \frac{1}{2\ell+1}\psi(x_0) + \frac{1}{3\sqrt{2\pi}}\beta x_0^{-\frac{1}{6}}.$$

Substituting these functions in formula (4), we find the corresponding expression for the sum S_2:

$$S_2 = -\frac{1}{36\pi}\frac{e^{j(\omega t - k^{(a)}r)}}{k^{(a)}r}\cos\phi\,\delta\varepsilon \left[x_0^{\frac{1}{3}}\left(\alpha + \beta x_0^{\frac{2}{3}}\right)^2 \sum_{\ell=\ell_1+1}^{\ell_2}(-1)^\ell - \frac{1}{4}\beta^2 x_0^{-\frac{1}{3}} \sum_{\ell=\ell_1+1}^{\ell_2}(-1)^\ell (2\ell+1)^2 \right]. \tag{9}$$

For $\ell + \frac{1}{2} \approx x_0$, the order of the expression in the brackets will depend essentially upon the quantity $x_0 = k^{(a)}R$, and the order of the entire sum S_2 will be determined by the product $\frac{R}{r}\delta\varepsilon$.

Finally, in the third region for $\ell + \frac{1}{2} > x_0$, the asymptotic formulas for cylindrical functions yield

$$\psi_\ell(x_0) = \frac{e^{jx_0 f_0}}{\sqrt{j \sin \tau_0}} \quad \text{and} \quad \psi_\ell'(x_0) = e^{jx_0 f_0}\sqrt{j \sin \tau_0}$$

If we set $\tau_0 = -j\,\theta_0$, then cosh $= \frac{\ell + \frac{1}{2}}{x_0}$ and $jf_0 = -\cosh\theta_0$ $(\theta_0 - \tanh\theta_0)$. For a large x_0 ($x_0 \gg 1$), the functions $\psi_\ell(x_0)$ and $\psi_\ell'(x_0)$ will decrease rapidly with an increase of θ_0.

The sum S_3 assumes the form

$$S_3 = -\frac{R}{r}\delta\varepsilon\cos\phi\, e^{j(\omega t - k^{(a)}r)}\sum_{\ell=\ell_2+1}^{\infty}(-1)^\ell \cos[\tau_0(\ell)]\, e^{2jx_0 f_0(\ell)}, \tag{10}$$

Grouping the components in pairs, we represent (as before) the latter sum (for even ℓ_2's) in the form

$$\sum_{\ell=\ell_2+1,\,\ell_2+3}^{\infty}\left[e^{-2j\tau_0(\ell)} - 1\right]\cos[\tau_0(\ell)]\, e^{2jx_0 f_0(\ell)}$$

If ℓ_2 is odd, the expression written will have a negative sign. As was shown previously, the problem reduces to the calculation of the sum

$$\sum_{\ell=\ell_2+1,\,\ell_2+2}^{\infty} F[\tau_0(\ell)]\, e^{2jx_0 f_0(\ell)} = \Phi(\infty) - \Phi(\ell_2+1),$$

where

$$F[\tau_0(\ell)] = -j\sin[\tau_0(\ell)],$$

$$\Phi(\ell) = \frac{1}{2}e^{2jx_0 f_0(\ell)}\, e^{j\tau_0(\ell)}, \quad \Phi(\infty) \to 0$$

$$\Delta\Phi(\ell) \approx F[\tau_0(\ell)]\, e^{2jx_0 f_0(\ell)}.$$

Finally, the formula for S_3 will have the form

$$S_3 = \frac{1}{2}\frac{R}{r}\delta\varepsilon\cos\phi\, e^{j(\omega t - k^{(a)}r)}\, e^{-2x_0\cosh\theta_0(\theta_0 - \tanh\theta_0) + \theta_0}, \tag{11}$$

where θ_0 is taken for the index $\ell = \ell_2 + 1$.

Here, as in the expressions for S_1 and S_2, we see the factor $\frac{R}{r}\delta\varepsilon$, which determines the order of these sums.

Formulas (5), (6), (8), (9), and (11) make it possible to calculate the sums S_1, S_2, and S_3 and to determine the field intensity of the diffracted wave

$$E_\theta = S_1 + S_2 + S_3 . \tag{12}$$

Later, the author plans to return to the solution of this problem by another method and to evaluate the error of the results obtained.

In conclusion, we note that the order of amplitude ($\frac{R}{r}\delta\varepsilon$) found for the electric field intensity may be obtained from elementary considerations. According to Fresnel's formula, the reflectance for normal incidence of the wave upon the surface of separation of two media is

$$\rho = \frac{I^{(r)'}}{I^{(e)}} = \left(\frac{n-1}{n+1}\right)^2 \approx \frac{\delta n^2}{4} .$$

Here $I^{(e)}$ and $I^{(r)'}$ respectively are the intensities of the incident and reflected waves and $\delta n = \frac{1}{2}\delta\varepsilon$ is the difference in the refractive indexes of the two media.

Let us consider now the scattering of a small electromagnetic beam by a spherical surface. If the cross-sectional area of the incident parallel beam is $\pi\Delta x^2$, after reflection its cross-sectional area at a distance r from the center of the sphere will increase to a value $\approx \pi\Delta x'^2$ and, correspondingly, the intensity of the reflected beam will be decreased: $I^{(r)'}:I^{(r)} = \Delta x'^2:\Delta x^2$, where $I^{(r)}$ is the intensity of the wave reflected by the sphere. It follows from simple geometry that $\frac{\Delta x'}{\Delta x} \approx \frac{2r}{R} - 1$ (neglecting infinitesimals, beginning with those of the second order). The ratio of the amplitudes of the reflected wave $E^{(r)}$ to the incident $E^{(e)}$ assumes the form

$$E^{(r)}:E^{(e)} = \frac{\delta n}{4}\,\frac{1}{\frac{r}{R}-\frac{1}{2}} = \frac{\delta\varepsilon}{8}\,\frac{1}{\frac{r}{R}-\frac{1}{2}} . \tag{13}$$

For $r \gg R$, the order of the ratio of the amplitudes $\frac{R}{r} \delta \varepsilon$ agrees with the result obtained previously.

More general results may also be obtained from simple considerations. Let us consider a dielectric bounded by an arbitrary convex surface having principal radii of curvature R_1 and R_2 which are large in comparison with the wavelength of the incident wave.

For normal incidence of a circular electromagnetic beam upon the surface of the dielectric, we have as before

$$\frac{\Delta x'}{\Delta x} \approx \frac{2r}{R} - 1 = \frac{2d}{R} + 1;$$

where R is the radius of curvature of the normal cross-section of the surface and $d = r - R$ is the distance (along the normal) from the surface to the point of observation. Using the Euler theorem

$$\frac{\Delta x'}{\Delta x} = 2d\left(\frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}\right) + 1,$$

we calculate (integrating with respect to θ) the cross-sectional area of the beam reflected by the surface.

Omitting the simple calculations, we find

$$P = \frac{I^{(r)'}}{I^{(r)}} = \frac{3d^2}{2}\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right) + \frac{d^2}{R_1 R_2} + 2d\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + 1, \quad (14)$$

and the ratio of the amplitudes desired assumes the form

$$\frac{E^{(r)}}{E^{(e)}} = \frac{\delta \varepsilon}{4\sqrt{P}} \quad (15)$$

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BIBLIOGRAPHY

1. Mie, G., Ann. d. Phys., (4), 25, 377, 1908.

2. Debye, P., Ann. d. Phys., (4), 30, 57, 1909.

3. Jobst, G., Ann. d. Phys., 76, 863, 1925; 78, 157, 1925.

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