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Translation

Stabilization of Space Vehicles

By

S.V. Cheremnykh

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STABILIZATION OF SPACE VEHICLES

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STABILIZATION OF SPACE VEHICLES -- NEW PROBLEMS AND METHODS

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[Text] A study is made of the problems of stabilizing spacecraft in the active segments of the flight from the point of view of some new methods of motion control theory.

A new approach to investigating the dynamic characteristics of a space vehicle as an object of control is discussed which is a development of the controllability and observability theory of Kalman as applied to the given class of objects.

The study is made of various problems in the analysis of the spacecraft dyanmics encountered in various planning and design stages.

The book is intended for engineering and technical workers involved in designing rockets and other flight vehicles.

FOREWORD

In this paper a discussion is presented of the methods of investigating the stabilizability of linear dynamic systems including an oscillating controlled target and controls of given structure as applied to the problems of controlling a spacecraft with a liquid-propulsion rocket engine.

It is known that liquid-propellant rockets are very difficult subjects for stabilization because of the unfavorable dynamic characteristics arising from the mobility of the fuel in the tanks, the elasticity of structure and also the nonsteady-state nature of the characteristics of the vehicle and the environment. Therefore in spite of using the latest methods of synthesizing control systems for such objects, frequently the optimal quality indexes which could be achieved are not achieved.

At the same time, in the design phase of the space vehicle as an object of control in practice there are always unused possibilities for selecting the structure and elements of the composite system basically determining its dynamic characteristics in the process of controlled movement.

The problem of how these possibilities can be used is the starting point for the studies, the results of which are discussed in this paper.

Of course, it would be desirable to solve the problem of optimizing the dynamic characteristics of a space vehicle in the most general form, considering the closed target-controller system as a whole, also taking into account the ballistic, strength and other requirements. For many reasons, primarily as a result of the "curse of size," as R. Bellman puts it, this is in practice impossible and against our wills it is necessary to limit ourselves to more modest goals.

Let us note that in the liquid-propulsion rocket design developments the situation is typical where the structure of the controls is rigidly given for one reason or another. For example, a spacecraft for a different purpose designed on the basis of some basic version can have the same automatic stabilization system with respect to structure with, perhaps, only the values of the parameters altered.

In such situations, when the structure of the controls is defined, it is natural to ask the following question: how are the structural parameters of the object of control selected so that it will have the best characteristics with respect to the controls of the given structure? To solve this problem it is desirable to have an optimalness criterion which does not depend on the specific parameters of the controls determined only by the structure of the controls and the characteristic parameters of the object. Formally, it is possible to construct such a criterion only after proving the theoretical possibility of separating the regions of stability and space of the parameters of the object and the controls (under assumptions that are reasonable for the investigated class of systems).

So far as the author knows, this was done for the first time in the papers by B. I. Rabinovich [57], I. M. Sidorov, I. P. Korotayeva [67] for controlled oscillatory systems with a controller which slightly disturbs the natural frequencies of the auxiliary oscillators.

There were other prerequisites for studying the problem of optimizing the dynamic properties of the object of control from general points of view. The fact is that in general control theory as a result of the work of R. Kalman and other researchers, there is a tendency at the present time to separate the investigation of the problems pertaining to the controlled system itself as the object of control into a separate region (the problems of controllability, observability [27], invariance [64], directivity [51], and so on).

Although the structure of the control system itself is completely ignored here, the corresponding criteria provide valuable information about the behavior of the object of control in the control process.

The methods discussed in this paper occupy an intermediate position between the corresponding methods of controllability (observability) theory and the classical theory of stability. The structure of the control system here is significant in contrast to the Kalman theory; at the same time the specific values of the parameters of the control system do not enter into various criterial relations, and the results of the investigations are formulated in terms of the regions in space of the parameters of the object of control itself.

In order to emphasize this fact, and also considering that the concept of stability is too overworked, in this book, following the lead of reference [56], we shall call the criteria characterizing the object of control the stabilizability criteria.

The idea of the proposed approach consists in the following. Some formal analog -- a quadratic form with coefficients which depend on the parameters of the object of control is placed in correspondence to this object. The positive (or negative) definability of this form is identified with the concept of perfection of the object (for example, the space vehicle) in the dynamic sense.

Testing the Sylvester conditions for the mentioned quadratic form and the construction of the corresponding regions in the space of the parameters of the object of control make up the content of the methods of investigating the stabilizability of controlled oscillatory systems.

If the terminology adopted in space vehicle dynamics is used, the discussed theory is the theory of "phase" stabilization of oscillatory systems. In practice when "phase" stabilization of the space vehicle is impossible, usually the problems of "amplitude" stabilization are investigated which are essentially the classical problems of the analysis of the stability of moving objects.

The methods of investigating stabilizability are to some degree analogous to the above-mentioned methods of investigating controllability, observability and so on in general control theory: both permit the general analysis of the properties of the object of control as the first step in solving the classical problems of stability of motion or various problems of optimal control.

Chapter 2 of this book contains a discussion of stabilizability theory.

Chapter 1 is an auxiliary chapter. The simplest model including two connected oscillators is used to investigate some of the problems characteristic of the modern theory of linear controlled systems (the problems of dynamic instability, controllability, observability, modal control in various situations, and so on).

Chapters 3 and 4 are of an applied nature. In these chapters a study is made of the problems of the stabilizability of various models of space vehicles and also adjacent problems pertaining to the design of optimal (in the dynamic sense) objects.

The mathematical models of space vehicles are used to the degree of completeness which corresponds to the level of the initial design phases of objects of this type: as a rule the equations are assumed to be linear, the coefficients are considered constant ("frozen" for some characteristic point in time τ of the active segment).

The oscillatory nature of the object of control in the given case comes from the presence of moving fuel components used to operate the sustainer engines and also elasticity of the hull and other structural elements.

Automatic stabilization systems (in the transverse oscillation mode) are used as the control systems here, and in the case of longitudinal oscillations, the engine is used directly. The performed studies of specific composite systems of space vehicles and the standard conditions of space vehicle movement provide a basis for considering that the discussed methods are a quite effective tool for investigating the dynamic properties of flight vehicles with liquid-propulsion rocket engines under the conditions of incompleteness of a priori information about the stabilization system. It is appropriate to emphasize here that the effectiveness of the methods of investigating the stabilizability is especially noticeable when they are used in combination with the classical methods of analysis, preceding them in the general process of investigating the stability of the investigated class of objects. Therefore, a discussion is presented below (see Chapter 5) of a number of the traditional methods of analyzing the stability of the closed system made up of the space vehicle and its control system, and the problems of amplitude stabilization are investigated. The author gives special attention here to the interpretation of the regions of stabilizability of space vehicles when investigating the stability of the control processes in the active segment.

On the whole, in this book the author would like to attract the attention of the readers to the new possibilities which are offered by successive (physical) analysis of such characteristics as controllability, observability, stabilizability, and so on as applied to dynamic systems of a large number of oscillatory degrees of freedom and with limited possibilities of modal control.

If we are talking about the general problem of stability, which in no way replaces the classical methods, this approach helps us to find the primary causes of instability and either to eliminate them or determine the direction of further research.

In conclusion, the author expresses his deep appreciation to doctor of technical sciences, Prof B. I. Rabinovich for valuable suggestions made when reviewing the manuscript of the book and also engineer Yu. V. Shchetinin for his assistance in preparing the manuscript for publication.

It is requested that all critical comments and suggestions be sent to the following address: Moscow, GSP-6, 1-y Basmannyy per., d.3, izd-vo "Mashinostroyeniye."

CHAPTER 1. SIMPLEST NONCONSERVATIVE OSCILLATORY SYSTEMS

1.1. Examples from Mechanics. Autonomous Systems with Two Degrees of Freedom

Aircraft Wing Flutter

Flutter is the classical example of a phenomenon which cannot be explained by investigating the system with only one degree of freedom. This form of instability is characterized by intensive interaction of at least two oscillatory elements under the effect of external nonpotential forces. The names given to the various forms of flutter illustrate this fact:

Bending-aileron wing flutter (bending vibrations of the wing combined with aileron vibrations);

Bending-rudder flutter of the horizontal empennage (bending vibrations of the fuselage in the vertical plane jointly with vibrations of the elevator around the axis of suspension);

Torsional-rudder flutter of the horizontal empennage (torsional vibrations of the fuselage combined with vibrations of the elevator and tail assembly), and so on.

This is how this phenomenon appears to observers from the outside [30]: "While testing an experimental aircraft, a twin-engine monoplane, the wings began to vibrate unexpectedly. This occurred while the aircraft was flying a measured base line at maximum speed near the ground. The wing vibrations began abruptly and were of an antisymmetric nature, that is, if the right half of the wing went up, the left half went down at the same time. Powerful vibrations of the ailerons occurred at the same time, so that the controls were jerked out of the pilot's hands. In the given case the pilot made the right decision: he throttled down and in spite of the fact that the controls had been jerked out of his hands, he suceeded in significantly taking up the elevator. As a result, the speed began to drop sharply, and when it had decreased by about 20%, the vibrations stopped almost as abruptly as they had begun. Five to eight seconds passed between the beginning and end of the vibrations. After the vibrations stopped, the aircraft behaved normally, and after 5 minutes of flight the pilot made a good landing at the airport.

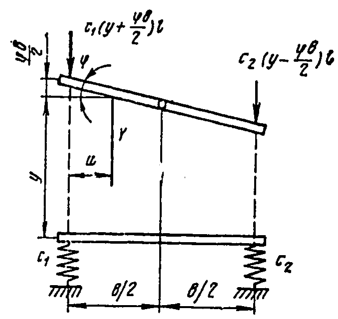


Figure 1.1. Force diagram on an inclined plate

"Examination of the aircraft revealed that at some points in the structure of the center wing section rupture of the skin and cowlings had started (cracks had formed), and some of the rivets holding the skin to the longerons, stringers and ribs had sheared off. Residual deformations in the form of waves in the skin had appeared on the surface of the center wing section between the engines and the fuselage. Clay was detected in the suspension of the ailerons and the aileron servotabs.

"However, flutter does not always end so favorably. Cases are known where vibrations that began in flight have led to complete disintegration of the aircraft in one or two seconds or less. It appears to observers of such an accident from the ground as if the part of the aircraft where the vibrations started has exploded."

Thus, observations indicate that under defined flight conditions vibrations of the fuselage and control surfaces which are extraordinarily intense can occur under defined flight conditions.

The complete theory of flutter is highly complex [9, 25, 76]. Here, only a suitable mechanical model [50] which can be used to explain the primary aspects of the nature of this phenomenon which once was a threatening obstacle on the path of increasing the speed of aircraft, will be investigated.

Let the plate depicted in Fig 1.1 have two degrees of freedom. We shall characterize its position by two coordinates -- the angle of rotation ϕ and vertical displacement y of the center of the plate.

Horizontal displacements will be considered impossible.

Let us write the equations defining ϕ and y as a function of time: $\phi=\phi(t)$; $y=y(t)$.

The tensions of the two springs c_1 and c_2 will be assumed to be different, and the mass of the plate, uniformly distributed over its entire surface. Let us denote by m the mass corresponding to a unit area of the central plane of the plate.

During its movement, the plate is acted on by the list

$$Y = \frac{dc_y}{d\alpha} \frac{\varrho V^2}{2} bl\varphi,$$

running at a distance a from the left edge of the plate and also the force of reaction of the elastic supports proportional to the displacements of the long sides of the plate:

$$R_1 = -\left(y + \frac{b}{2}\varphi\right) c_1 l; \; R_2 = -\left(y - \frac{b}{2}\varphi\right) c_2 l.$$

Reducing these reactions to the center of gravity of the plate, we obtain the force

$$R = R_1 + R_2 = -(c_1 + c_2) ly - \frac{b}{2}(c_1 - c_2) l\varphi$$

and the moment

$$M = \left(y + \frac{b}{2}\varphi\right) c_1 l \frac{b}{2} + \left(y - \frac{b}{2}\varphi\right) c_2 l \frac{b}{2}.$$

Now let us select the equations of motion of the plate. One of them describes the center of the gravity of the plate:

$$Y + R = mbl \frac{d^2y}{dt^2}, \quad (1.1)$$

where mbl is the mass of the entire plate and the other, the rotations of the plate around the horizontal axis z passing through the center of mass:

$$Y\left(\frac{b}{2} - a\right) + M = \frac{mb^2 l}{2} \frac{d^2\varphi}{dt^2}. \quad (1.2)$$

Substituting the expressions for Y, R and M in equations (1.1) and (1.2), we obtain the following system of differential equations:

$$\frac{d^2y}{dt^2} + a_{11}y + a_{12}\varphi = 0; \quad (1.3)$$

$$\frac{d^2\varphi}{dt^2} + a_{21}y + a_{22}\varphi = 0,$$

where

$$a_{11}=\frac{c_1+c_2}{mb};\quad a_{22}=\frac{3(c_1+c_2)}{mb}-\frac{dc_4}{d\alpha}\varrho\frac{v^2}{2}\frac{b-2a}{mb^2};$$

$$a_{12}=\frac{c_1-c_2}{2m}-\frac{dc_y}{d\alpha}\varrho\frac{v^2}{2}\frac{1}{m};\quad a_{21}=\frac{6(c_1-c_2)}{mb^2}. \qquad (1.4)$$

When solving the problems of aeroelasticity, to which the discussed topic belongs, first the conditions are defined to which the static form of the stability loss, the so-called divergence, is possible.

For this purpose it is proposed that y and ϕ are constants. Then the second derivatives vanish and the equations assume the form

$$a_{11}y+a_{12}\varphi=0,\ a_{21}y+a_{22}\varphi=0. \qquad (1.5)$$

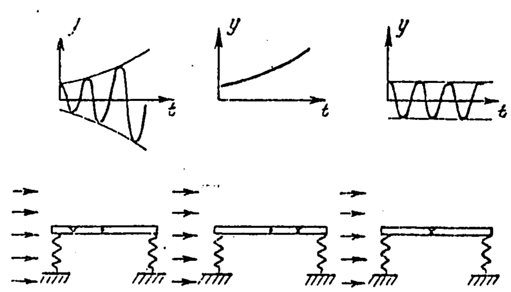
The condition of nonzero solutions of system (1.5) has the form

$$a_{11}a_{22}-a_{12}a_{21}=0.$$

Substituting the expressions (1.4) for the coefficients, the following formula is obtained, which defines the critical divergence rate:

$$v_{кр}^{(1)}=2\sqrt{\frac{c_1}{\varrho\frac{dc_y}{d\alpha}\left(1-\frac{c_1}{c_2}\right)}}. \qquad (1.6)$$

Key: 1. critical



a b c

Figure 1.2. Types of movements of the plate after initial deflection:
the axis of rigidity is denoted by the x, the center of mass of the plate is in the middle of the span.

Divergence as a static form of loss of stability is possible obviously when $c_1<c_2$. Limiting ourselves by this remark, let us proceed to the investigation of other possible solutions of systems (1.3) as an oscillatory system.

We shall find the solution to y(t), ϕ(t) in the form

$$y(t)=Ae^{i\omega t},\ \varphi(t)=Be^{i\omega t}, \qquad (1.7)$$

in other words, we propose that both coordinates vary according to the same law and they are always proportional to each other.

It is easy to see that the desired system is satisfied by values of both ω and ($-\omega$). Therefore for the coordinates y, ϕ the solution will be found in the form, respectively

$$y=A_1e^{i\omega t}+A_2e^{-i\omega t}, \qquad (1.8)$$
$$\varphi=B_1e^{i\omega t}+B_2e^{-i\omega t}. \qquad (1.8b)$$

Three cases can be represented:

1) ω is the real number, for example $\omega=\theta$. Then, using the Euler formula $e^{i\theta}=\cos\theta+i\sin\theta$, we find

$$y=A_1(\cos\theta t+i\sin\theta t)+A_2(\cos\theta t-i\sin\theta t)=\tilde{A}_1\cos\theta t+\tilde{A}_2\sin\theta t,$$

where $\tilde{A}=A_1+A_2;\ \tilde{A}_2=\frac{A_1-A_2}{i}$ are the new constants.

The expressions (1.8, a), (1.8, b) are harmonic movement (Fig 1.2, c). In this case the state of equilibrium obviously is stable.

2) ω is a complex number, for example $\omega=i\beta$. Then

$$y=Ae^{-\beta t}+A_2e^{\beta t}.$$

This solution (see the second term) is unlimited and increases monotonically. In this case the state of equilibrium must be recognized as unstable (Fig 1.2, b).

3) Let ω be a complex number, for example, $\omega=\alpha+i\beta$. Then the solution is the function

$$y=A_1e^{i\alpha t}e^{-\beta t}+A_2e^{-i\alpha t}e^{\beta t}.$$

Whatever the number β, positive or negative, the solution of y(t), and ϕ(t) together with it, will increase without limit. The factors $e^{i\omega t}$, $e^{-i\omega t}$ indicate oscillations of increasing amplitude. This condition is flutter.

From the point of view of applied problems, of course, the nature of the buildup of the amplitudes y(t) and ϕ(t) is of interest, especially at the beginning of the process. This problem, however, will be considered somewhat later, and for the time being we shall return to the system (1.3).

Substituting expressions (1.7) for y(t) and ϕ(t) in it, we find

$$A(-\omega^2+a_{11})+Ba_{12}=0,\ Aa_{21}+B(-\omega^2+a_{22})=0.$$

The determinate of this system must equal zero as a condition of existence of the nonzero values of A and B:

$$\begin{vmatrix} \omega^2+a_{11} & a_{12} \\ a_{21} & -\omega^2+a_{22} \end{vmatrix}=0,$$

that is,

$$\omega^4-\omega^2=(a_{11}+a_{22})+(a_{11}a_{22}-a_{12}a_{21})=0.$$

Hence, we find

$$\omega^2=\frac{a_{11}+a_{22}}{2}\pm\sqrt{\left(\frac{a_{11}+a_{22}}{2}\right)^2-(a_{11}a_{22}-a_{12}a_{21})}.$$

The condition of reality of ω^2 acquires the form

$$a_{11}a_{22}-a_{12}a_{21}\leqslant\left(\frac{a_{11}+a_{22}}{2}\right)^2, \tag{1.9}$$

and the condition of positiveness of ω^2:

$$a_{11}a_{22}-a_{12}a_{21}\leqslant 0. \tag{1.10}$$

Thus, for stability of the investigated mechanical system it is necessary that

$$0\leqslant a_{11}a_{22}-a_{12}a_{21}\leqslant[(a_{11}+a_{22})/2]^2.$$

It is obvious that the boundaries of this interval correspond to the critical states

$$a_{11}a_{22}-a_{12}a_{21}=0; \tag{1.11}$$

$$a_{11}a_{22}-a_{12}a_{21}=\left(\frac{a_{11}+a_{22}}{2}\right)^2. \tag{1.12}$$

The first equality was investigated above as the condition of divergence, and it led to the formula (1.6). In this case the frequencies

$$\omega_1^2 = a_{11} + a_{12}; \; \omega^2 = 0.$$

The second equality also determines the instability as the constant deviation of the system (static instability). Actually, for the smallest violation of the condition (1.10), one of the values of ω^2 will be obtained with a minus sign. This corresponds to aperiodic motion illustrated in Fig 1.2, b. Divergence is possible under the condition $c_1 < c_2$. For $c_1 > c_2$ the condition is always satisfied and it remains only to analyze the equality (1.12). Substituting the expressions for the coefficients (1.4) in it, we find the critical velocity

$$v_{\text{кр}(1)} = \sqrt{\frac{1}{\frac{dc_y}{d\alpha}\varrho}\left[(c_1 - c_2) + \frac{m^2 b^2 (a_{11} - a_{22})^2}{3(c_1 - c_2)}\right]}. \qquad (1.13)$$

Key: 1. critical

With the slightest increase in the velocity v above the critical value $v_{critical}$, ω^2 becomes a complex number, and the movement of the plate acquires the nature depicted in Fig 1.2, b. Thus, formula (1.13) determines the speed of onset of flutter.

The presented arguments explain (at least qualitatively) the picture of flutter described in the beginning of this section. Obviously, at some point the speed of the aircraft was close to critical and by changing the flight conditions the pilot was able to reduce it before the vibrations completely destroyed the aircraft.

Returning to formula (1.13), we note that for $v_{critical}$ the corresponding frequencies of the system ω_1^2, ω_2^2 coincide and numerically become equal to $\omega_1^2 = \omega_2^2 = \frac{(a_{11} + a_{22})}{2}$. This fact also serves as a necessary condition of the occurrence of classical flutter.

Let us proceed to other examples indicating the possibility of the analogous effect also under other conditions.

Double Pendulum Loaded Under a Following Load

Let us consider the model shown in Fig 1.3 [82]. Let ℓ be the length of the rigid element; m_1, m_2 be concentrated masses at distances a_1, a_2, respectively, from the ends of the rods; c be the rigidity of the elastic hinges; P be the tracking force.

The Lagrange equations will be taken in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \frac{\partial v}{\partial q_i} + \tilde{Q} \; (i = 1, 2), \qquad (1.14)$$

where T, v are the kinetic and potential energies respectively; Q are the nonpotential generalized forces.

Assuming the usual simplifications arising from smallness of the oscillations, we find for the given case

$$T=\frac{1}{2}\left[m_1a_1^2\dot{\theta}_1^2+m_2(l\dot{\theta}_1+a_2\dot{\theta}_2)^2\right];$$
$$v=\frac{c}{2}\left[\theta_1^2+(\theta_2-\theta_1)^2\right]-\frac{Pl}{2}(\theta_1^2+\theta_2^2); \qquad (1.15)$$
$$\tilde{Q}_1=\tilde{Q}_2=-Pl\theta_2.$$

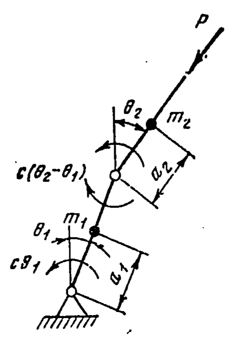


Figure 1.3. Model of a double pendulum under the effect of a following load ($m_1=m_2$)

The differential equations of motion (1.14) assume the following form as a result of substitution of expressions (1.15)

$$(m_1a_1^2+m_2l^2)\ddot{\theta}_1+m_2la_2\ddot{\theta}_2+(2c-Pl)\theta_1-(c-$$
$$-Pl)\theta_2=0;\ m_2la^2\ddot{\theta}+m_2a_2^2\ddot{\theta}_2-c\theta_1+c_2\theta_2=0.$$
$$(1.16)$$

Finding the solution of (1.16) in the form $\theta_1=Ae^{pt}$, $\theta_2=Be^{pt}$, we obtain with respect to p the characteristic equation

$$a_0p^4+ap^2+a_4=0 \qquad (1.17)$$

with the coefficients

$$a_0=m_1m_2a_1^2a_2^2;$$
$$a_2=\left[m_1a_1^2+m_2a_2^2+m_2(l+a_2)^2\right]c-m_2a_2(l+a_2)Pl;$$
$$a_4=c^2.$$

Since $a_4>0$, there is no static instability of the type of the above-investigated divergence here.

Let us investigated equation (1.17). For simplification let us set

$$m_1=m_2=\frac{m}{2};\quad a_1=a_2=\frac{l}{2};$$

then

$$a_0=\frac{1}{64}\,m^2l^4;\quad a_2=-\frac{1}{8}\,ml^2(11c-3Pl);\quad a_4=l^2,$$

and the discriminant of the characteristic equation

$$\Delta=a_2^2-4a_0a_4=\frac{3}{64}\,m^2l^4(39c^2-22cPl+3P^2l^2).$$

The condition of instability, just as in the preceding problem, has the form $\Delta<0$ or

$$3P^2l^2-22cPl+39c^2<0, \tag{1.18}$$

which imposes defined conditions on the magnitude of the following load P.

Solving the inequality (1.18), we find

$$3\frac{c}{l}<P<\frac{13}{3}\,\frac{c}{l}. \tag{1.19}$$

This situation is illustrated by Fig 1.4. For $P<3c/\ell$ the roots $P^2_{1,2}$ are negative, and the pendulum is stable as a result. In the interval (1.19) the roots will be complexly conjugate, and for $P>(\frac{13}{3})\frac{c}{\ell}$ the discriminant Δ is positive, but the coefficient a_2 is negative. Hence, it follows that the roots $P^2_{1,2}$ are positive. In both cases the pendulum is unstable. Thus, the critical value of the load is the value

$$P_{\text{кр (1)}}=3\frac{c}{l}. \tag{1.20}$$

Key: 1. critical

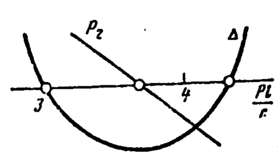


Figure 1.4. The functions $\Delta(P)$ and $a_2(P)$ determining the stability of a double pendulum

For $P>3c/\ell$ the pendulum, being disturbed, begins to undergo oscillations analogous to the oscillations in the flutter problem (see Fig 1.2, b). For $P=P_{critical}$ the natural values of the system -- as solutions of the characteristic equation (1.10) -- coincide and for the given case assume the form

$$\omega_1^2=\omega_2^2=-\frac{1}{16}ml^2(11c-3P_{\text{кр}}l).$$
(1)

Key: 1. critical

Electric Circuit with Oscillator

Let us consider the circuit depicted in Fig 1.5 which is made up of two circuits (L_1c_1), (L_2c_2) connected by a high capacitance c [65]. As the generalized coordinates let us take $q_1(t)$, $q_2(t)$ -- the charges on the capacitors c_1 and c_2 respectively.

The equations of the electromagnetic oscillations in the circuit are obtained from the Lagrange equations (1.14), where

$T=\frac{L_1\dot{q}_1^2}{2}+\frac{L_2\dot{q}_2^2}{2}$ is the magnetic energy of the system and

$U=\frac{q_1^2}{2c_1}+\frac{q_2^2}{2c_2}+\frac{q_1q_2}{c}$ is the electrostatic energy of the system. After performing the required operations we obtain

$$L_1\frac{d^2q_1}{dt^2}+\left(\frac{1}{c_1}+\frac{1}{c}\right)q_1-\frac{1}{c}q_2=0;$$
$$L_2\frac{d^2q_2}{dt^2}+\left(\frac{1}{c_2}+\frac{1}{c}\right)q-\frac{1}{c}q_1=0. \qquad (1.21)$$

These two equations describe the variation in time of the charges q_1 and q_2 on the plates of the capacitors c_1 and c_2 at the given time. It is obvious that these equations are a strict analogy of the equations which describe the movement of two harmonic oscillators joined to each other by a spring.

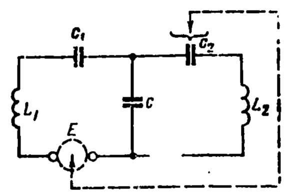


Figure 1.5. Connected electric circuits (the auxiliary circuit simulating the following load is depicted by the dotted line)

Let us introduce an oscillator into the circuit which does not have internal resistance. It is possible to measure the potential difference on the terminals, for example, of the capacitor c_2 and use it for adjustment of E by installing amplifier tubes so that E will be proportional to this voltage:

$$E=\lambda q_2\left(\frac{1}{c_2}+\frac{1}{c}\right),$$

or for simplification of notation:

$$E=\beta=\frac{q}{c}.$$

Now the system of equations which describes the oscillations in the circuit assumes the following form:

$$L_1\frac{d^2q_1}{dt^2}+\left(\frac{1}{c_1}+\frac{1}{c}\right)q-\frac{1+\beta}{c}q_2=0; \qquad (1.22)$$

$$L_2\frac{d^2q_2}{dt^2}+\left(\frac{1}{c^2}+\frac{1}{c}\right)q_2-\frac{1}{c}q_1=0.$$

The characteristic equation of system (1.22) has the form

$$\begin{vmatrix} L_1p^2+\left(\frac{1}{c_1}+\frac{1}{c}\right) & -\frac{1+\beta}{c} \\ -\frac{1}{c} & L_2p^2+\left(\frac{1}{c^2}+\frac{1}{c}\right) \end{vmatrix}=0$$

or

$$L_1L_2p^4+\left[L_1\left(\frac{1}{c^2}+\frac{1}{c}\right)+L_2\left(\frac{1}{c_1}+\frac{1}{c}\right)\right]p^2+$$
$$+\left(\frac{1}{c_1}+\frac{1}{c}\right)\left(\frac{1}{c^2}+\frac{1}{c}\right)-\frac{1+\beta}{c^2}=0. \qquad (1.23)$$

The discriminant of the equation (1.23)

$$D=\left[L_1\left(\frac{1}{c_2}+\frac{1}{c}\right)+L_2\left(\frac{1}{c_1}+\frac{1}{c}\right)^2+\frac{4(1+\beta)}{c^2}L_1L_2=0\right] \quad (1.24)$$

becomes negative if $\beta<\beta_{critical}$,

where

$$\beta_{кр}^{(1)}=-1-\frac{c^2}{4L_1L_2}\left[L_1\left(\frac{1}{c^2}+\frac{1}{c}\right)-L_2\left(\frac{1}{c_1}+\frac{1}{c}\right)\right]^2. \quad (1.25)$$

Key: 1. critical

From expressions (1.24) and (1.25) it follows that if the parameter β is positive and larger than one, the natural frequencies of the system

$$\omega_{1,2}^2=\frac{1}{2}\left[L_1\left(\frac{1}{c_2}+\frac{1}{c}\right)+L_2\left(\frac{1}{c_1}+\frac{1}{c}\right)\right]\pm$$
$$\pm\frac{1}{2}\sqrt{\left[L_1\left(\frac{1}{c_2}+\frac{1}{c}\right)+L_2\left(\frac{1}{c_1}+\frac{1}{c}\right)\right]+\frac{4(1+\beta)}{c^2}L_1L_2}$$

move apart still more with respect to their values for $\beta=0$.

If the parameter β is negative, the asymmetric relation in equation (1.22) creates an effect trying to bring the natural frequencies together.

For $\beta=\beta_{cr}$ the frequencies

$$\omega_1^2=\omega_2^2=\frac{1}{2}\left[L_1\left(\frac{1}{c_2}+\frac{1}{c}\right)+L_2\left(\frac{1}{c_1}+\frac{1}{c}\right)\right]$$

merge, and with a further decrease in β, instability of the same nature as in the above-investigated examples occurs.

Thus, as is obvious from what has been discussed, in all cases where an asymmetric relation is added to a symmetric (or Lagrangian) relation which is realized as a result of the presence of quadratic terms in the expressions for the kinetic (magnetic) and potential (electrostatic) energy, new, different effects are possible. The nature of the instability which can arise accordingly differs from the known effect of "negative" friction or resonance (under the effect of an external periodic force).

The principal necessary attribute of this from of instability is approach of the natural frequencies of the system to merging of them.

How this influences the properties of the system as an object of control and what the role of such factors as dissipative forces is will be investigated below.

1.2. Oscillations of Unstable Systems

Terminological Remarks

The above-investigated systems obviously were not conservative. However, the definition of an "unconservative system" is in need of more precise statement, considering that there are different points of view in this regard. For example, this is how "unconservativeness" is interpreted in the book by T. Karman and M. Biot [29]: "The term 'unconservative' strictly speaking refers to the case of so-called dissipative systems in which the mechanical or electrical energy is converted to thermal energy. In certain other cases the unconservative nature of the forces is caused to a known degree by the method of investigation. For example, the system made up of the air foil and flap is an unconservative system in the sense that it can absorb energy or give it to the environment -- the air. The entire system, including the air, the wing and the flap, if we neglect viscosity, is a conservative system. Thus, the conservative or unconservative nature of the system depends on how our mechanical system is isolated."

It is known that a system is called conservative if: 1) the system is scleronomic; 2) all the forces are potential forces; 3) the potential energy U does not explicitly depend on time.

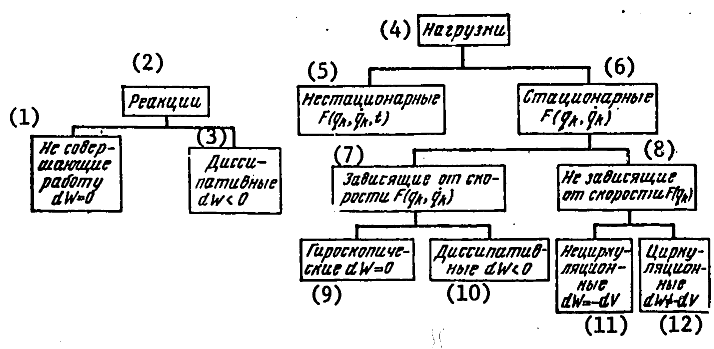


Figure 1.6. Classification of reactions and loads

Key:

1. Not completing work dW=0
2. Reactions
3. Dissipative dW<0
4. Loads
5. Nonstationary $F(q_k, \dot{q}_k, t)$
6. Stationary $F(q_k, \dot{q}_k)$
7. Depending on velocity $F(q_k, \dot{q}_k)$
8. Not depending on velocity $F(q_k)$
9. Gyroscopic dW=0
10. Dissipative dW<0
11. Noncirculating dW=dV
12. Circulating dw≠dV

For a conservative system the work performed by external forces depends only its initial and final position. For conservative systems there is an energy integral dE/dt=0 (the total energy of the system does not change during its movement).

As for systems combined under the term of "nonconservative," in this paper the classification of T. Ziegler [82] will be adopted. The corresponding table is presented in Fig 1.6 where q_k, $\dot{q}_k$ are the generalized coordinates and velocities of the system; dW is the elementary work of the force F in the segment of actual displacement dq_k; $U(q_k)$ is the potential.

Let us call the system unconservative if it contains at least one unconservative force, that is, a load of the dissipative ($F=F(\dot{q}_k)$, $dW<0$) or circulating ($F=F(q_k)$, $dW \neq -dU$) type. (Nonstationary loads are not considered in this paper.)

In the examples of Section 1 the dissipative forces were absent. The aerodynamic and following forces were used as an example of circulating forces. In the future we shall talk also about controlling forces which depend both on the generalized coordinates and on the generalized velocities and on the basis of the adopted terminology are a special case of "dissipative" and "circulating" forces.

Equations of Motion. General Solution in the Case of Instability

If, in addition to the potential forces defined by the function U nonpotential forces also act on the system

$$\tilde{Q}_i = \tilde{Q}_i(t, q_i, \dot{q}_i) \quad (i=1, 2, \ldots, n),$$

the Lagrange equations assume the form (1.14). As is obvious from the examples, the general procedure when rating the equations consists in the following.

Let us consider the potential energy U, which is a function of the generalized coordinates $q_1, q_2, \ldots, q_n$ and also contains the additive constant:

$$U = U(c_1 q_1, \ldots, q_n). \qquad (1.26)$$

Let us take the system of coordinates so that the position of equilibrium, with respect to which the movement is investigated, will have the coordinates $q_1=q_2= \ldots =q_n=0$.

The constant c will be determined from the condition $U(c, 0, \ldots, 0)=0$.

Let us assume that in the position of equilibrium of the system, the necessary conditions of stability are satisfied:

$$\frac{\partial U}{\partial q_1}=\frac{\partial U}{\partial q_2}=\ldots=\frac{\partial U}{\partial q_n}=0.$$

In this case the function $U(q_1,\ldots,q_n)$, being expanded in a Maclaurin series at the point $(0,\ldots,0)$:

$$U=U(0,\ldots,0)+\sum_{i=1}^{n}\left(\frac{\partial U}{\partial q_i}\right)_{q_i=0}q_i+\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 U}{\partial q_i\partial q_k}q_iq_k+\ldots,$$

in the first approximation is represented in the form of a quadratic form of the generalized coordinates $q_1, q_2,\ldots,q_n$:

$$U=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 U}{\partial q_i\partial q_k}q_iq_k=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}k_{ik}q_iq_k, \qquad (1.27)$$

where $k_{ik}=k_{ki}$ on the basis of the known properties of displaced derivatives of the functions U. In the position of stable equilibrium the quadratic form of U is positively defined, that is, it does not assume a negative value for any values of the variables.

The potential forces $Q_1,\ldots,Q_n$ acting on the system in an arbitrary position $q_1,\ldots,q_n$, are obtained by differentiation of the relations (1.27):

$$\begin{aligned}-Q_1&=\frac{\partial U}{\partial q_1}=k_{11}q_1+k_{12}q_2+\ldots+k_{1n}q_n;\\-Q_2&=\frac{\partial U}{\partial q_2}=k_{21}q_1+k_{22}q_2+\ldots+k_{2n}q_n;\\&\ldots\ldots\ldots\ldots\ldots\ldots\\-Q_n&=\frac{\partial U}{\partial q_n}=k_{n1}q_1+k_{n2}q_2+\ldots+k_{nn}q_n.\end{aligned} \qquad (1.28)$$

On the basis of symmetry the system (1.28) has defined properties. Let us multiply each of the equations by $q_1,\ldots,q_n$ respectively and add. Then

$$-\sum_{i=1}^{n}Q_iq_i=\sum_{i=1}^{n}\frac{\partial U}{\partial q_i}=\sum_{i=1}^{n}\sum_{k=1}^{n}k_{ik}q_iq_k=2U$$

or

$$-\frac{1}{2}\sum_{i=1}^{n} Q_i q_i = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial U}{\partial q_i} q_i = U.$$

This relation expresses the fact that if the system goes under the effect of the external forces $-Q_1, \ldots, -Q_n$ from the position of equilibrium to an arbitrary position $q_1, \ldots, q_n$, its potential energy increases by the amount equal to the work of the forces which accomplished this displacement.

Hence, it follows that for a conservative system (we are still not considering the forces $\tilde{Q}_1, \ldots, \tilde{Q}_n$) the work of the external forces in the closed cycle (that is, the process where the values of the generalized coordinates at the beginning and end of the process are equal) will be identically equal to zero.

In general it is necessary to note that the condition $k_{ik} = k_{ki}$ is a necessary and sufficient condition of conservativeness of the system having potential energy of the type (1.27).

Let us introduce the kinetic energy of the system

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j, \qquad (1.29)$$

into the investigation, which is a definitely positive quadratic form of the variables $\dot{q}_i$, $\dot{q}_j$. Substituting expressions (1.29) and (1.27) in the Lagrange equations (1.14), we obtain the equations of the oscillations in the form

$$M\ddot{q} + Kq = 0, \qquad (1.30)$$

where M and K are the symmetric matrices,

$$M = \|m_{ij}\|_1^n;\ K = \|k_{ij}\|_1^n;\ q = (q_1, \ldots, q_n).$$

Then as the nonpotential forces $\tilde{Q}_i$, the circulating forces of the type will be taken into account

$$\tilde{Q}_i = -\sum_{j=1}^{n}\tilde{k}_{ij}q_j, \qquad (1.31)$$

where the coefficients $\tilde{k}_{ij}$ and $\tilde{k}_{ji}$ (at least for one pair) are not equal to each other, and the generalized forces which are the sum of the dissipative and gyroscopic forces are:

$$\tilde{Q}_i = -\sum_{j=1}^{n} \tilde{k}_{ij} q_j, \qquad (1.32)$$

where the coefficients $\tilde{k}_{ij}$ and $\tilde{k}_{ji}$ can be arbitrary.

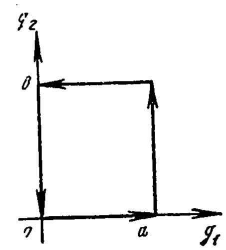


Figure 1.7. Closed cycle in the q_1, q_2 plane

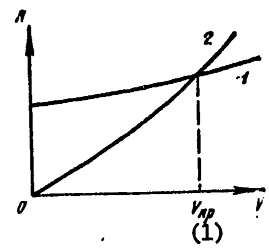


Figure 1.8. Qualitative dependence of the work of the nonpotential forces on the flight speed:
1 -- damping forces; 2 -- disturbing forces
Key: 1. $V_{critical}$

Substituting the expressions (1.31) and (1.32) in the Lagrange equation (1.14), we obtain the system

$$M\ddot{q} + B\dot{q} + \tilde{K}q = 0,$$

a special case of which is the systems investigated in the preceding section and also investigated below.

What does the asymmetry of the matrix $\tilde{k}$ generated by the presence of nonpotential forces Q_i lead to (from the physical point of view)? In the case where n=2

$$-\tilde{Q}_1 = \tilde{k}_{11} q_1 + \tilde{k}_{12} q_2; \quad -Q_2 = \tilde{k}_{21} q_1 + \tilde{k}_{22} q_2$$

are external forces calculated considering the expression (1.31) so that $\tilde{k}_{12} \neq \tilde{k}_{21}$.

Let us calculate the work of the external forces in a closed cycle defined in the plane (q_1, q_2) (Fig 1.7) by the following equations:

$$q_1 = 0; \; q_2 = 0; \; q_1 = a; \; q_2 = b.$$

We obtain

$$W = -\int_0^a \tilde{k}_{11} q_1 dq_1 - \int_0^b (\tilde{k}_{21} q_1 + \tilde{k}_{22} q_2)\, dq_2 - \int_a^0 (\tilde{k}_{11} q_1 + \tilde{k}_{12} q_2)\, dq_1 - \int_b^a \tilde{k}_{22} q_2 dq_2 = (\tilde{k}_{12} - \tilde{k}_{21})\, ab. \quad (1.33)$$

If $\tilde{k}_{12}-\tilde{k}_{21}$ 0, the work of the external forces is positive, that is, during the course of the indicated process the system will receive additional energy.

Thus, the asymmetry of the equations of the system creates conditions under which the system can exhaust the energy from the environment, which is exhibited in an increase in the amplitude of the oscillations of the generalized coordinates.

In the investigated examples such conditions occurred when the parameters v, P, β exceeded some critical values: $v > v_{critical}$; $P > P_{cr}$; $\beta > \beta_{cr}$.

Consideration of the dissipative forces excluded from these arguments complicates the calculations of the critical values of the parameter. In general, exact analysis of both the circulating and dissipative forces makes the corresponding problem far from elementary. From the physical point of view in many cases the situation is quite clear, as is obvious, for example, in the problem of flutter [30]: "As a result of the work of the exciting forces, an increase in amplitude of the oscillations takes place which in the final analysis can lead to fracture of the structure. As a result of the damping forces, the amplitude of the oscillations decreases, for the energy of the vibrating system is spent on overcoming the damping forces, and if the damping effects are sufficiently powerful, the amplitude decreases to zero, that is, the oscillations are damped. The aerodynamic forces, both exciting and damping, depend on the flight speed, and, consequently, the work performed by them depends on the flight speed. However, with an increase in the speed, the exciting forces can increase faster than the damping forces, and beginning with some flight speed, the work performed by exciting forces begins to exceed the work expended by the vibrating system on overcoming the damping effects. Beginning with this flight speed, the oscillation amplitude under the effect of any random pulse will increase, that is, the flutter increases. Fig 1.8 shows the approximate variation in work of the exciting and damping forces and one oscillation cycle with an increase in flight speed. If the flight speed is less than critical, then the oscillations that occur damp; if it is greater, then it takes place with increasing amplitude."

Thus, the presence of circulating forces of the type of (1.31) leads to asymmetry of the system of oscillation equations, asymmetry (on the formal side) leads to new effects which make the problem of stability and, later, the problems of control meaningful.

Let us consider an "asymmetric" model:

$$\ddot{q}_1+q_1+\beta_1 q_2=0,$$
$$\ddot{q}_2+\alpha q_2+\alpha\beta_2 q_1=0; \qquad (1.34)$$

here $q_1(t)$, $q_2(t)$ are the generalized coordinates, α, β_1, β_2 are the parameters.

Let us propose that the characteristic equation of the system

$$p^4+p^2(1+\alpha)+(\alpha-\alpha\beta_1\beta_2)=0 \qquad (1.35)$$

has the roots

$$p_{1,2}=\delta\pm i\omega;\ p_{3,4}=-\delta\pm i\omega, \qquad (1.36)$$

indicating its instability.

Considering expressions (1.36) it is possible to transform the system (1.34), excluding one parameter:

$$\ddot{y}_1+(\omega^2-\delta^2)y_1+2\delta\omega y_2=0,$$
$$\ddot{y}_2+(\omega^2-\delta^2)y_2-2\delta\omega y_1=0. \qquad (1.37)$$

The characteristic equation

$$p^4+2(\omega^2-\delta^2)p^2+(\delta^2+\omega^2)^2=0. \qquad (1.38)$$

varies correspondingly.

Let us investigate the process of the development of the oscillations as a result of some initial disturbance.

The obvious symmetry of roots (1.36) compensates to some degree for the comparatively high order of the initial system and permits in practice an entire analysis in general form.

In accordance with the general theory let us set

$$q_1(t)=Ae^{\delta t}\cos(\omega t+\varphi_1)+Be^{-\delta t}\cos(\omega t+\varphi_2), \qquad (1.39)$$
$$q_2(t)=A\varrho e^{\delta t}\cos(\omega t+\varphi_1+\varkappa)+B\varrho e^{-\delta t}\cos(\omega t+\varphi_2-\varkappa). \qquad (1.40)$$

In formulas (1.39)-(1.40) the constants A, B, ϕ_1, ϕ_2 are determined by the initial conditions q_{10}, q_{20}, $\dot{q}_{10}$, $\dot{q}_{20}$, and the parameters ρ, χ, by the properties of the system itself, and they are calculated from the expressions

$$\sin\varkappa=\frac{2\delta\omega}{\delta^2+\omega^2};\quad \cos\varkappa=\frac{\delta^2-\omega^2}{\delta^2+\omega^2};\ \varrho=\delta^2+\omega^2. \qquad (1.41)$$

The solution in the form of (1.39)-(1.40) has little suitability for studying the nature of the oscillations on the basis of the implicit dependence of the constants A, B, ϕ_1, ϕ_2 on the initial conditions. Let us perform some transformations.

It is possible to consider the functions $q_1(t)$, $q_2(t)$ as the results of the addition of two oscillations of identical frequency ω with different phases and amplitudes.

Using the expression

$$a\cos x + b\sin x = \sqrt{a^2+b^2}\cos(x-\theta),$$

where

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}};\quad \cos\theta = \frac{a}{\sqrt{a^2+b^2}},$$

let us transform formulas (1.39)-(1.40):

$$q_1(t) = M(t)\cos(\omega t + \psi_1(t)); \qquad (1.42)$$

$$q_2(t) = N(t)\cos(\omega t + \psi_2(t)). \qquad (1.43)$$

The envelopes M(t), N(t) and the "phases" $\psi_1(t)$, $\psi_2(t)$ are calculated from the expression

$$M(t) = \sqrt{A^2e^{2\delta t} + B^2e^{-2\delta t} + 2AB\cos(\varphi_1-\varphi_2)}; \qquad (1.44)$$

$$N(t) = \varrho\sqrt{A^2e^{2\delta t} + B^2e^{-2\delta t} + 2AB\cos(\varphi_1-\varphi_2+2\varkappa)}; \qquad (1.45)$$

$$\begin{cases} \cos\psi_1 = \dfrac{1}{M(t)}\left[Ae\cos\varphi_1 + Be^{-\delta t}\cos\varphi_2\right]; \\ \cos\psi_2 = \dfrac{1}{N(t)}\left[Ae^{\delta t}\cos(\varphi_1+\varkappa) + Be^{-\delta t}\cos(\varphi_2-\varkappa)\right]; \\ \sin\psi_1 = \dfrac{1}{M(t)}\left[Ae^{\delta t}\sin\varphi_1 + Be^{-\delta t}\sin\varphi_2\right]; \\ \sin\psi_2 = \dfrac{1}{N(t)}\left[Ae^{\delta t}\sin(\varphi_1+\varkappa) + Be^{-\delta t}\sin(\varphi_2-\varkappa)\right]. \end{cases}$$

A further step consists in calculating the explicit dependence of the integration constants A, B, ϕ_1, ϕ_2 on the initial conditions q_{10}, q_{20} $\dot{q}_{10}$, $\dot{q}_{20}$.

The system of equations for defining the constants A, B, ϕ_1, ϕ_2 is obtained by the usual method -- by calculating the functions (1.39)-(1.40), and also their first derivatives at the initial point in time t=0:

$$\begin{aligned} &A\cos\varphi_1+B\cos\varphi_2=q_{10};\\ &A\cos(\varphi_1+\varkappa)+B\cos(\varphi_2-\varkappa)=\frac{q_{20}}{\varrho};\\ &A\delta\cos\varphi_1-A\omega\sin\varphi_1-B\delta\cos\varphi_2-B\omega\sin\varphi_2=\dot q_{10}; \qquad (1.46)\\ &A\delta\varrho\cos(\varphi_1+\varkappa)-A\omega\varrho\sin(\varphi_1+\varkappa)-B\varrho\delta\cos(\varphi_2-\varkappa)-B\varrho\omega(\varphi_2-\varkappa)=\frac{\dot q_{20}}{\varrho}.\end{aligned}$$

The relations (1.46) can be simplified by carrying out trigonometric transformations and separating the terms containing the unknowns. We obtain

$$\begin{aligned} &A\cos\varphi_1+B\cos\varphi_2=f_1;\\ &-A\sin\varphi_1+B\sin\varphi_2=f_2; \qquad (1.47)\\ &A\sin\varphi_1+B\sin\varphi_2=f_3;\\ &A\cos\varphi_1-B\cos\varphi_2=f_4.\end{aligned}$$

The righthand sides of the system (1.47) are calculated by the formula

$$\begin{aligned} &f_1=q_{10};\\ &f_2=-q_{10}\operatorname{ctg}\varkappa+\frac{1}{2\delta\omega}q_{20};\\ &f_3=2\delta\omega(\delta\cos\varkappa-\omega\sin\varkappa)\dot q_{10}-\frac{2\delta^2\omega}{\delta^2+\omega^2}\dot q_{20};\\ &f_4=2\delta\omega(\delta\sin\varkappa+\omega\cos\varkappa)\dot q_{10}-\frac{2\delta\omega}{\delta^2+\omega^2}\dot q_{20}\end{aligned}$$

and they are, as is obvious, linear combinations of the initial conditions q_{10}, q_{20}, $\dot q_{10}$, $\dot q_{20}$ with coefficients that depend on the parameters δ, ω, otherwise determined by the properties of the system itself.

The system (1.47) decays into two independent subsystems, on the solution of which it is possible to find the required relations:

$$A=\frac{1}{2}\sqrt{(f_1+f_4)^2+(f_3-f_2)^2};\quad B=\frac{1}{2}\sqrt{(f_1-f_4)^2+(f_3+f_2)^2}; \qquad (1.48)$$

$$\begin{cases}\sin\varphi_1=\frac{1}{2A}(f_3-f_4);\\ \cos\varphi_1=\frac{1}{2A}(f_1+f_4)\end{cases}\qquad \begin{cases}\sin\varphi_2=\frac{1}{2B}(f_3+f_4);\\ \cos\varphi_2=\frac{1}{2B}(f_1-f_4).\end{cases}$$

In order to obtain the final result it is necessary to substitute expressions (1.48) in the formulas (1.44)-(1.45). As a result, we obtain the following formulas for the envelopes M(t), N(t):

$$M(t)=\frac{1}{\sqrt{2}}\sqrt{F_{0M}\operatorname{ch}(2\delta t)+F_{1M}\operatorname{sh}(2\delta t)+F_{2M}};$$
$$N(t)=\frac{\varrho}{\sqrt{2}}\sqrt{F_{0N}\operatorname{ch}(2\delta t)+F_{1N}\operatorname{sh}(2\delta t)+F_{2N}}, \quad (1.49)$$

where

$$F_{0M}=F_{0N}=f_1^2+f_2^2+f_3^2+f_4^2;\ F_{1M}=F_{1N}=2(f_1f_4-f_2f_3);$$
$$F_{2M}=f_1^2-f_2^2+f_3^2-f_4^2;\ F_{2N}=(f_1^2-f_2^2+f_3^2-f_4^2)\cos 2\varkappa+ \quad (1.50)$$
$$+2(f_1f_2+f_3f_4)\sin 2\varkappa,$$
$$\operatorname{ch}(2\delta t)=\frac{1}{2}(e^{2\delta t}+e^{-2\delta t});\quad \operatorname{sh}(2\delta t)=\frac{1}{2}(e^{2\delta t}-e^{-2\delta t}).$$

The "phases" ψ_1, ψ_2 are calculated from the expression

$$\operatorname{tg}\psi_1=\frac{f_3-f_2\operatorname{th}(\delta t)}{f_1+f_4\operatorname{th}(\delta t)};\quad \operatorname{tg}\psi_2=\frac{(f_3+f_4\operatorname{tg}\varkappa)+(-f_2+f_1\operatorname{tg}\varkappa)\operatorname{th}(\delta t))}{(f_1+f_2\operatorname{tg}\varkappa)+(f_4-f_3\operatorname{tg}\varkappa)\operatorname{th}(\delta t)}, \quad (1.51)$$

with the exception of the cases where $\cos\psi_1$, $\cos\psi_2$ vanish.

Thus, the solution of the initial system (1.37) has the form

$$q_1(t)=M(t)\cos(\omega t+\psi_1);\quad q_2=N(t)\cos(\omega t+\psi_2),$$

where the envelopes M(t), N(t) and the phases ψ_1, ψ_2 are defined by the expressions (1.50), (1.51) in the form of the functions of the system parameters and the initial conditions.

Study of the Nature of the Transient Process

It is known that if the characteristic equation of the system has the roots $p_{1,2}=\pm i\omega_1$, $p_{3,4}=\pm i\omega_2$, the general solution of the system has the form

$$q_1(t)=A\cos(\omega_1 t+\alpha_1)+B\cos(\omega_2 t+\alpha_2);$$
$$q_2(t)=A\varrho_1\cos(\omega_1 t+\alpha_2)+B\varrho_2\cos(\omega_2 t+\alpha_2).$$

The physical meaning of these relations consists in the fact that each of the coordinates q_1, q_2 generally speaking completes the sum of two harmonic oscillations with different normal frequencies ω_1, ω_2. These frequencies are given by the system itself. The system also gives the ratios ρ_1, ρ_2 of the amplitudes of each normal oscillation in both coordinates. The amplitudes and phases are given by the initial conditions. It is significant that each of the harmonic oscillations has identical phase in both coordinates.

In the case of dynamic instability of the same system the situation is different.

The relations (1.42)-(1.43) can be interpreted as the complex oscillations of the generalized coordinates $q_1(t)$, $q_2(t)$ with phases and amplitudes that vary in time.

What is new here by comparison with conservative systems is that the phase shift between the oscillations q_1 and q_2 is given not only by the initial conditions, but also the system itself (the parameter χ), and, in addition, it depends on time.

The sharply expressed increasing nature of the oscillations is also a theoretically new effect. It is clear that the source of energy here is the external force proportional to q_i and "encoded" in the coefficients β_1, β_2.

Let us consider how the instability develops in the system taken out of the position of equilibrium by the initial impetus. Let us set $\delta t \ll 1$, $\delta \ll \omega$.

In formulas (1.49)-(1.50) we shall limit ourselves to the terms of the second order of smallness with respect to (δt):

$$\mathrm{sh}(2\delta t)\approx 2\delta t;\quad \mathrm{ch}(2\delta t)\approx 1+2\delta^2 t^2;\quad \mathrm{th}(\delta t)\approx \delta t.$$

After simple transformations we find:

$$\begin{aligned} M(t)&=\sqrt{\frac{F_{0M}+F_{2M}}{2}}\left[1+\frac{F_{1M}}{F_{0M}+F_{2M}}(\delta t)+\frac{F_{0M}}{F_{0M}+F_{2M}}(\delta t)^2\right];\\ N(t)&=\sqrt{\frac{F_{0N}+F_{2N}}{2}}\left[1+\frac{F_{1N}}{F_{0N}+F_{2N}}(\delta t)+\frac{F_{0N}}{F_{0N}+F_{2N}}(\delta t)^2\right]. \end{aligned} \tag{1.52}$$

Then in formulas (1.50) let us set $\delta \ll \omega$ and let us estimate the order of the functions $f_i (i=1,\dots, 4)$ with respect to the parameter δ. We obtain

$$f_3=\delta^2\tilde{f}_3,\ f_4=\delta\tilde{f}_4,\ f_2=\frac{1}{\delta}\tilde{f}_2,\ f_1=\tilde{f}_2, \tag{1.53}$$

where $\lim\limits_{\delta\to 0}\frac{\delta}{\tilde{f}_i}=0\ (i=1,\dots,4).$

The expressions (1.52) are taken in the form

$$M(t)=a_M+b_M t+c_M t^2;\ N(t)=a_N+b_N t+c_N t^2, \tag{1.54}$$

where

$$a_M=\frac{1}{\sqrt{2}}\sqrt{F_{0M}+F_{2M}};\ b_M=\frac{1}{\sqrt{2}}\ \frac{F_{1M}}{\sqrt{F_{0M}+F_{2M}}};$$

$$c_M=\frac{1}{\sqrt{2}}\ \frac{F_{0M}}{\sqrt{F_{0M}+F_{2M}}};$$

$$a_N=\frac{1}{\sqrt{2}}\sqrt{F_{0N}+F_{2N}};\ b_N=\frac{1}{\sqrt{2}}\ \frac{F_{1N}}{\sqrt{F_{0N}+F_{2N}}};\ c_N=\frac{1}{\sqrt{2}}\ \frac{F_{0N}}{\sqrt{F_{0N}+F_{2N}}}. \tag{1.55}$$

Let $F_{1N}=F_{1M}$. Then in the first approximation the oscillation envelopes M(t), N(t) are approximated by straight lines

$$M(t)=a_M+b_Mt;\ N(t)=a_N+b_Nt. \qquad (1.56)$$

Let us estimate the dependence of the angular coefficients b_N, b_M on the parameter δ. Considering (1.53) we have

$$\begin{aligned} F_{1N}=F_{1M}=2(f_1f_4-f_2f_3)=\delta\tilde{F}^{(1)};\\ \sqrt{F_{0M}+F_{2M}}=\sqrt{2}\sqrt{f_1^2+f_2^2}=\tilde{F}^{(2)};\\ \sqrt{F_{0N}+F_{2N}}=\sqrt{2}\sqrt{(f_1\cos x+f_2\sin x)^2+(f_3\cos x+f_4\sin x)^2}, \end{aligned} \qquad (1.57)$$

where $\tilde{F}^{(k)}$ (k=1,2,3) are constants that have the order 1 by comparison with the value of δ.

Substituting expressions (1.57) in formulas (1.56), we find

$$b_M=\delta\frac{\tilde{F}^{(1)}}{\tilde{F}^{(2)}},\ b_N=\delta\frac{\tilde{F}^{(1)}}{\tilde{F}^{(2)}}. \qquad (1.58)$$

Thus, the envelopes M(t), N(t) are approximated for $\delta t \ll 1$ by straight lines, the angular coefficients of which are proportional to the degree of instability (δ) of the system.

There is a set of initial conditions favorable in some respects for which

$$f_1f_4-f_2f_3=0. \qquad (1.59)$$

Then the coefficients $F_{1M}=F_{1N}=0$, and the envelopes M(t), N(t) are approximated by the parabolas

$$M(t)=a_M+c_Mt^2;\ N(t)=a_N+c_Nt^2,$$

which for small δt give smaller increments of the oscillations than straight lines.

Then let us consider the problem of the relative oscillation phases of the generalized coordinates q_1, q_2.

For simplification let us set

$$q_{10}=\dot{q}_{20}=0,\ q_{10}\neq 0,\ q_{20}\neq 0.$$

Then on the basis of (1.47) $f_3=0$, $f_4=0$ considering $\delta \ll \omega$, we obtain the expressions

$$\operatorname{tg}\psi_1 = -\frac{f_2}{f_1}\operatorname{tg}(\delta t) \approx -\delta\frac{f_2}{f_1}t;$$
$$\operatorname{tg}\psi_2 = -\frac{f_2 - f_1\operatorname{tg}\varkappa}{f_1 + f_2\operatorname{tg}\varkappa}\operatorname{tg}(\delta t) \approx -\delta\frac{f_2 - f_1\operatorname{tg}\varkappa}{f_1 + f_2\operatorname{tg}\varkappa}t.$$

For the small angles $\operatorname{tg}\psi_1(t) \approx \psi_1(t)$; $\operatorname{tg}\psi_2(t) \approx \psi_2(t)$ and, consequently,

$$\psi_1(t) \approx -\left(\delta\frac{f_2}{f_1}\right)t;\ \psi_2 = -\delta\left(\frac{f_2 - f_1\operatorname{tg}\varkappa}{f_1 + f_2\operatorname{tg}\varkappa}\right)t.$$

Let us denote

$$\Delta\omega_1 = -\delta\frac{f_2}{f_1};\ \Delta\omega = -\delta\frac{f_2 - f_1\operatorname{tg}\varkappa}{f_1 + f_2\operatorname{tg}\varkappa}.$$

Then the solution to system (1.37) under the given assumptions is approximated by the functions

$$q_1(t) = (a_M + b_M t)\cos[(\omega + \Delta\omega_1)t];$$
$$q_2(t) = (a_N + b_N t)\cos[(\omega + \Delta\omega_2)t]. \quad (1.60)$$

The physical meaning of the formulas (1.60) consists in the fact that the system (1.37) behaves in the initial period of time just as if it were made up of two unconnected oscillatory elements under the effect of periodic forces under resonance conditions.

In the general case of arbitrary initial conditions the phase function $\psi_1(t)$, $\psi_2(t)$ is more complex.

In conclusion, let us consider a numerical example having the purpose of illustrating the process of the development of the oscillations for the case of commensurate values of the parameters (δ, ω).

Let δ=0.5; ω=2.0 (the system characteristic) and q_{10}=0.2; q_{20}=0.2; $\dot{q}_{10}+\dot{q}_{20}=0$ (initial conditions).

In this case, by calculating the necessary coefficients in the formulas (1.44) and (1.45), we find

$$M(t) = 0{,}2576\sqrt{e^t + e^{-t} - 1{,}4};$$
$$N(t) = 1{,}0948\sqrt{e^t + e^{-t} - 1{,}966}. \quad (1.61)$$

The following intermediate results were used in expressions (1.61):

$$\sin\varkappa = 0{,}4706;\ \cos\varkappa = -0{,}8823;\ \varkappa = 151{,}9°;\ \varphi_1 = -\varphi_2 = -67{,}2°;\ \varrho = 4{,}25;$$
$$A = B = 0{,}2576.$$

The angles ψ_1, ψ_2 for each of the points in time are determined from the equations

$$\operatorname{tg}\psi_1 = -2{,}375\ \operatorname{th}\left(\frac{t}{2}\right);\ \operatorname{tg}\psi_2 = -10{,}802\ \operatorname{th}\left(\frac{t}{2}\right).$$

The graphs of the functions $q_1(t)$, $q_2(t)$ considering the above-presented data are presented in Figure 1.9, a. In Fig 1.9, b there are graphs of the functions $q_1(t)$, $q_2(t)$ for another set of initial conditions: $q_{10}=q_{20}=0$; $\dot{q}_{10}=\dot{q}_{20}=0.2$.

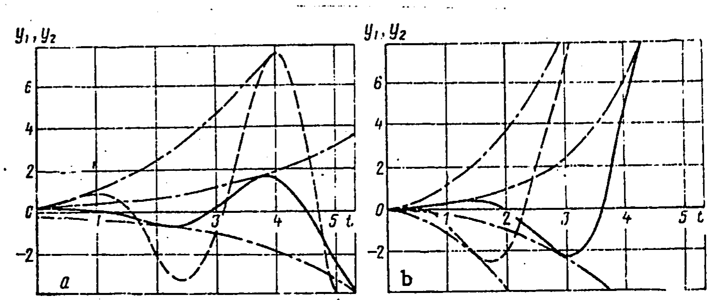


Figure 1.9. Nature of disturbed motion of a double pendulum

In this case

$$M(t) = 0{,}125\sqrt{e^t - e^{-t} - 0{,}560};$$
$$N(t) = 0{,}531\sqrt{e^t + e^{-t} - 1{,}906}.$$

The performed analysis indicates that even with a small degree of instability in a short time interval the oscillations acquire large amplitude, which is inadmissible in technical systems. Therefore the analysis of the factors limiting the buildup of the oscillations acquires important significance. In the following section a study is made of the effect of the dissipative forces.

1.3. Effect of Dissipative Forces on Stability

Root Hodographs Near the Stability Boundary System

Let us introduce the dissipative forces into the investigation. The control system (1.37) assumes the form

$$\ddot{q}_1+\gamma_1\dot{q}_1+q_1+\beta_1 q_2=0;$$
$$\ddot{q}_2+\gamma_2\dot{q}_2+\alpha q_2+\beta_2 q_1=0. \tag{1.62}$$

The characteristic equation of system (1.62)

$$(p^2+\gamma_1 p+1)(p^2+\gamma_2 p+\alpha)-\alpha\beta_1\beta_2=0 \tag{1.63}$$

is written in convenient form for application of the root hodograph method [8]:

$$\Phi_4(p)+k\Psi(p)=0 \tag{1.64}$$

or

$$\prod_{\nu=1}^{4}(p-p_\nu)+k\prod_{\mu=1}^{3}(p-z_\mu)=0,$$

where k is a variable parameter ($-\infty<k<\infty$); p_ν are the initial points of the root trajectories (k=0); z_μ are the limiting points of the root trajectories ($k\to\pm\infty$).

Setting $p=\delta+i\omega$ in equation (1.63) and successively assigning values to k in the indicated interval, we obtain the equation of the plane curve

$$f(\omega,\delta)=0,$$

called the root hodograph or the trajectory of the roots of the linear system. Let us consider a number of cases, taking the values of γ_1 and γ_2 as the parameters.

The stability boundary of the system (1.62) in the absence of dissipative forces is defined by the equality

$$(1-\alpha)^2+4\alpha\beta_1\beta_2=0. \tag{1.65}$$

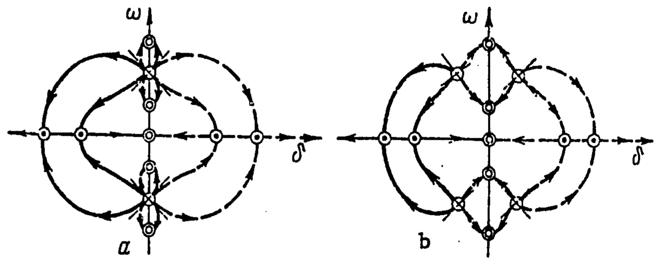


Figure 1.10. Root hodographs for a double pendulum: parameters -- damping coefficients γ_1, γ_2

Considering expression (1.65) the characteristic equation (1.63) is written in the form (γ_2=0)

$$p^4+(1+\alpha)p^2+\frac{(1+\alpha)^2}{4}+\gamma_1(p^2+\alpha)=0 \qquad (1.66)$$

or (for γ_1=0) in the form

$$p^4+(1+\gamma)p^2+\frac{(1+\alpha)^2}{4}+\gamma_2 p(p^2+1)=0. \qquad (1.67)$$

Calculating the characteristic points of the root hodographs corresponding to the equations (1.66)-(1.67), we obtain the picture (Fig 1.10, a) where the solid line corresponds to the positive values of γ_i (i=1, 2), and the dotted line, negative values.

(For determinacy we set α=1.44).

From an analysis of the root hodographs it follows that if the system is at the boundary of the dynamic instability (in the absence of damping), the damping of one of the partial systems cannot make it stable. On the contrary, in the given case the damping is a destabilizing factor.

Let us consider the more general case of $\gamma_1 \neq 0$, $\gamma_2 \neq 0$. The characteristic equation of the system has the form

$$p^4+(1+\alpha)p^2+\frac{(1+\alpha)^2}{4}+\gamma_2 p(p^2+1)+\gamma_1 p(p^2+\gamma_2 p+\alpha)=0. \quad (1.68)$$

As is obvious, the initial points p_ν (for γ_1=0) for the equation (1.67) depend on the parameter γ_2 and lie on the dotted curve of Fig 1.11. The standard root trajectories for γ_1>0 (the parameter γ_2, which defines the initial and the limiting points, is fixed) are presented in Fig 1.11. In this figure a study is made of the most interesting region -- in the vicinity of the imaginary access, for small values of the damping coefficients. As is obvious, fixing the values of the parameter γ_2 and assigning a number of values to the parameter γ_1, it is possible to make the system both stable and unstable. The destabilizing effect is expressed most clearly for small values of the parameter γ_2 (or, on the contrary, the parameter γ_1).

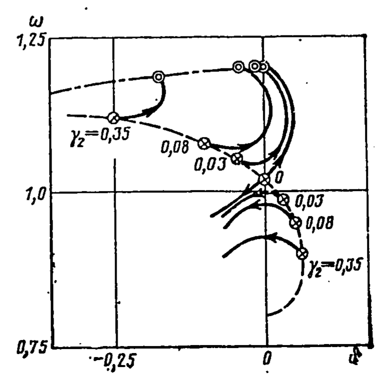


Figure 1.11. Root hodographs for a double pendulum near the stability boundary: parameters -- damping coefficients γ_1, γ_2

Let us consider the regions of stability of the system (1.62) for arbitrary values of the parameters γ_i, i=1,2. Let us write the characteristic equation (1.63) considering (1.65) in the form

$$p^4+(\gamma_1+\gamma_2)p^3+(1+\alpha+\gamma_1\gamma_2)p^2+(\gamma_1\alpha+\gamma_2)p+\frac{(1+\alpha)^2}{4}=0. \quad (1.69)$$

Since all of the coefficients of this equation are positive, then (by the Liénard-Schipard criterion) the only condition of stability of the system (1.62) has the form

$$\Delta_3>0,$$

where

$$\Delta_3=\gamma_2\gamma_1^2\alpha+\left[\gamma_2^2(1+\alpha)-\frac{(1-\alpha)^2}{4}\right]\gamma_1^2+\left[\gamma_2^3+\gamma_2\frac{(1-\alpha)^2}{2}\right]-\gamma_2^2\frac{(1-\alpha)^2}{4}. \quad (1.70)$$

The stability boundary of the regions of the system in the plane (γ_1, γ_2) is given by the equation $\Delta_3(\gamma_1, \gamma_2)=0$, and in the general case it has the form indicated in Fig 1.12. Here it is demonstrated, in particular, in what cases there are one or more transitions through the boundary of the region of stability on variation of one of the parameters γ_1. The best version of the relation between the damping coefficients from the point of view of insuring maximum stability reserves of the system corresponds obviously to the case of $\gamma_1=\gamma_2$.

The last conclusion also follows directly from the expression $\Delta_3=2(1+\alpha)\gamma^4\geq 0$, where $\gamma_1=\gamma_2=\gamma$.

Now let the following condition be satisfied:

$$(1-\alpha)^2+4\alpha\beta_1\beta_2<0$$

for instability of the system (1.62) (in the absence of dissipative forces).

Let us denote $\tilde{\chi}=-(1-\alpha)^2-4\alpha\beta_1\beta_2>0$, and let us transform the characteristic equation (1.63) to the form

$$p^4+(\gamma_1+\gamma_2)p^3+(1+\alpha+\gamma_1\gamma_2)p^2+(\gamma_2+\gamma_1\alpha)p+\left[\frac{\tilde{x}+(1+\alpha)^2}{4}\right]=0. \quad (1.71)$$

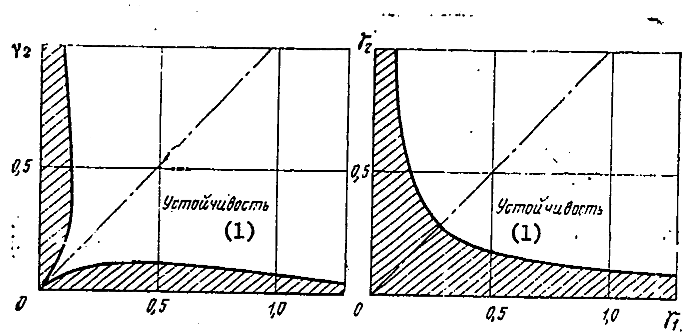


Figure 1.12. Regions of stability of a double pendulum in the plane of the parameters γ_1, γ_2

Key:
1. Stability

The equation (1.62) corresponds to the Hurwitz inequality

$$\Delta_3=\alpha\gamma_1^3\gamma_2+(1+\alpha)\gamma_1^2\gamma_2^2+\gamma_1\gamma_2^3-\frac{(1-\alpha)}{4}(\gamma_2-\gamma_1)^2-\frac{\tilde{x}(\gamma_1+\gamma_2)^2}{4}>0.$$

The regions of stability constructed in the parameters γ_1, γ_2 are presented in Fig 1.12.

In the given case the root hodographs of the system can be investigated on the basis of equation (1.71). In particular, setting $\gamma_1=0$ or $\gamma_2=0$, we obtain the picture indicated in Fig 1.10, b, where, just as before, the solid lines correspond to the positive values of the parameter γ, and the dotted lines, to the negative values. As is obvious, the damping of the oscillations of one of the oscillators will not be a stabilizing factor if the system is unstable in the absence of dissipative forces.

Now let us return to the specific mechanical system as applied to which the problem of the effect of dissipative forces on stability will be considered in more detail.

Double Pendulum: Destabilizing Damping Effect

Again let us consider a double pendulum loaded under a following force [82]. Let us alter the pendulum system somewhat (see Fig 1.13), and let us set $m_1=2m$; $m_2=m$. Let us introduce dissipative forces proportional to the generalized velocities $\dot{\phi}_i$ $(i=1,2)$.

The kinetic energy T, the dissipative function D, the potential energy V and the generalized forces Q_1 and Q_2 are defined by the expressions

$$T=\frac{1}{2}ml^2(3\varphi_1^2+2\varphi_1\varphi_2+\varphi_2^2);\quad D=\frac{1}{2}b_1\dot{\varphi}_1^2+\frac{1}{2}b_2(\dot{\varphi}_1^2-2\dot{\varphi}_1\dot{\varphi}_2+\dot{\varphi}_2^2);$$
$$V=\frac{1}{2}c(2\varphi_1^2-2\varphi_1\varphi_2+\varphi_2^2);\quad Q_1=Pl(\varphi_1-\varphi_2),\ Q_2=0. \qquad (1.72)$$

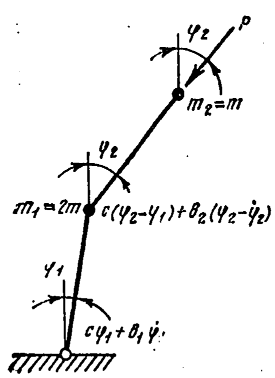


Figure 1.13. Model of a double pendulum under the effect of a following load and considering dissipative forces

From the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\varphi}_i}\right)+\frac{\partial D}{\partial\dot{\varphi}_i}-\frac{\partial T}{\partial\dot{\varphi}_i}+\frac{\partial V}{\partial\varphi_i}=$$
$$=Q_i,\ i=1,2, \qquad (1.73)$$

we obtain the linear equations of motion

$$3ml^2\ddot{\varphi}_1+(b_1+b_2)\dot{\varphi}_1-(Pl-2c)\varphi_1+ml^2\ddot{\varphi}_2-b_2\dot{\varphi}_2+Pl-c)\varphi_2=0;$$
$$ml^2\ddot{\varphi}_1-b_2\dot{\varphi}_1-c\varphi_1+ml^2\ddot{\varphi}_2+b_2\dot{\varphi}_2+c\varphi_2=0. \quad (1.74)$$

Let us assume that the solutions of system (1.74) have the form

$$\varphi_i=Ae^{p_i t}, \quad i=1,2; \quad p=i\omega. \quad (1.75)$$

Substituting expressions (1.75) in the system (1.74), we obtain the characteristic equation

$$a_0\Omega^4+a_1\Omega^3+a_2\Omega^2+a_3\Omega+a_4=0 \quad (1.76)$$

with the coefficients $a_0=a$; $a_1=B_1+6B_2$; $a_2=7-2F+B_1B_2$; $a_3=B_1+B_2$; $a_4=1$.

Dimensionless parameters are used in equation (1.76)

$$\Omega=l\omega\sqrt{\frac{m}{c}}; \quad B_i=\frac{B_i}{l\sqrt{cm}}; \quad i=1,2; \quad F=\frac{Pl}{c}. \quad (1.77)$$

In the absence of damping ($B_1=B_2=0$), the characteristic equation is biquadratic:

$$2\Omega^4+(7-2F)\Omega^2+1. \quad (1.78)$$

In this case the roots Ω_j (j=1,2,3,4) have the form

$$\Omega=\frac{1}{2}\left\{\pm\sqrt{F-\left(\frac{7}{2}-\sqrt{2}\right)}\pm\sqrt{F-\left(\frac{7}{2}+\sqrt{2}\right)}\right\} \quad (1.79)$$

and depending on the values of F they can be purely imaginary, complex or real.

In the graphs of Fig 1.4, the roots are represented by the points of intersection of the corresponding curves with the horizontal plane which is perpendicular to the F-axis and passes through the point corresponding to the given value of F.

The axonometric representation of the graphs of the variation of the roots and the rectangular projections of them on the real (Im Ω=0), imaginary (Re Ω=0) and complex (F=0) planes are presented also in this figure.

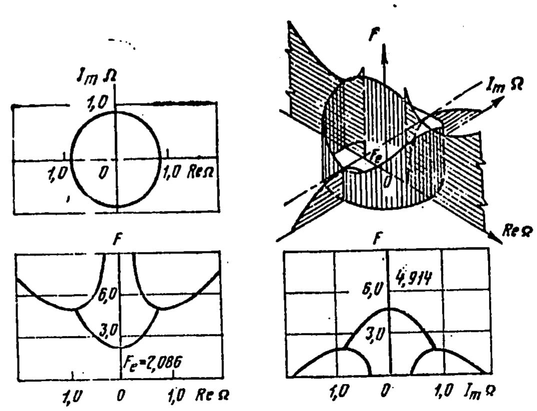


Figure 1.14. Rectangular projections and axonometric representation of the roots of the characteristic equation in the absence of damping

As is obvious, there are always two roots with positive real part if

$$F > \frac{7}{2} - \sqrt{2} = 2{,}086 = F_e.$$

For $F=F_e$ there are two pairs of equal roots, the real parts of which are equal to zero. Thus, the system is unstable for $F \geq F_e$. For $F<F_e$ all the roots are different and purely imaginary. Consequently, in this case the system is stable.

Let us consider the system with small damping, setting $B_1=B_2=0.01$.

There are no simple expressions for the roots of the characteristic equation (1.76) for $B_1 \neq 0$, $B_2 \neq 0$.

The numerical results obtained are presented in Fig 1.15 where three projections on the same three planes as in Fig 1.15 are presented to supplement the axonometric representation. Two roots will have positive real parts for $F>1.464=F_d$.

The stability of the system (1.74) can be investigated directly without determining the roots of the characteristic equation by applying the Routh-Hurwitz criterion.

In the given case the system is stable if

$$a_2 = 2\left[-F + \frac{1}{2}(7 + B_1B_2)\right] > 0;$$

$$\Delta_2 = 2(B_1 + 6B_2)\left\{-F + \left[\frac{5(B_1 + 8B_2)}{2(B_1 + 6B_2)} + \frac{1}{2}B_1B_2\right]\right\} > 0; \qquad (1.80)$$

$$\Delta_3 = 2(B_1^2 + 7B_1B_2 + 6B_2^2)\left\{-F + \left[\frac{4B_1^2 + 33B_1B_2 + 4B_2^2}{2(B_1^2 + 7B_1B_2 + 6B_2^2)} + \frac{1}{2}B_1B_2\right]\right\} > 0.$$

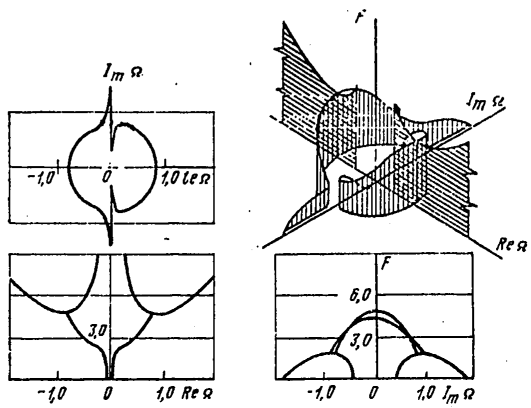


Figure 1.15. Rectangular projections and axonometric representation of the roots of the characteristic equation in the presence of small damping

In order that the system be stable, the value must satisfy the following free inequalities:

$$F < \frac{1}{2} + \frac{1}{2}B_1B_2; \quad F < \frac{5(\beta + 8)}{2(\beta + 6)} + \frac{1}{2}B_1B_2;$$

$$F < \frac{4\beta^2 + 33\beta + 4}{2(\beta^2 + 7\beta + 6)} + \frac{1}{2}B_1B_2, \qquad (1.81)$$

where $\beta = \frac{B_1}{B_2}; \; 0 \leqslant \beta \leqslant \infty.$

Inasmuch as

$$\frac{5(\beta + 8)}{2(\beta + 6)} = \frac{5}{2} + \frac{5}{\beta + 6} \leqslant \frac{10}{3} < \frac{7}{2}$$

and

$$\frac{4\beta^2 + 33\beta + 4}{2(\beta^2 + 7\beta + 6)} = \frac{5(\beta + 8)}{2(\beta + 6)} - \frac{\beta + 3}{2(\beta + 1)} < \frac{5(\beta + 8)}{2(\beta + 6)} < \frac{7}{2},$$

for any value of β in its range of variation it is obvious that the critical load is determined by a third inequality, that is,

$$\overline{F}_d<0,\ F_d=\frac{4\beta^2+33\beta+4}{2(\beta^2+7\beta+6)}+\frac{1}{2}B_1B_2.$$

The value of F_d thus depends on the ratio of the damping coefficients and on each of them individually.

For B_i<<1, just as for infinitely small damping, $\overline{F}_d$ assumes the value

$$\overline{F}_d=\frac{4\beta^2+33\beta+4}{2(\beta^2+7\beta+6)},$$

which essentially depends on β, being, generally speaking, smaller than F_e and never exceeding it. Depending on β, the ratio F_d/F_e was represented graphically in Fig 1.16.

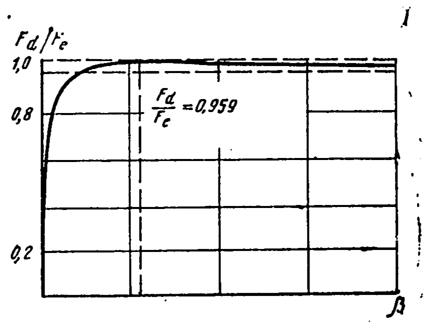


Figure 1.16. Critical load as a function of the damping coefficient

It is possible to see that for $\beta=4+5\sqrt{2}\approx 11.07$, the ratio F_d/F_e reaches a maximum value of 1.

Thus, in the given special case the destabilizing effect is excluded.

For β=0 the ratio F_d/F_e reaches maximum value equal to 0.16, that is, in the investigated system with two degrees of freedom the maximum destabilizing effect is about 84%.

1.4. Controllability of the Oscillatory System with Two Degrees of Freedom

Controllability and Observability Criteria

Let us introduce a controllable system

$$\begin{aligned}\ddot{q}_1+q_1+\beta_1q_2&=u,\\ \ddot{q}_2+\alpha q_2+\alpha\beta_2q_1&=cu,\end{aligned}\qquad(1.82)$$

into the investigation which is distinguished from an "uncontrollable" system by the presence of a controlling function u(t) in the righthand side (C is a parameter).

Let us assume that on formation of the feedback circuit in the system (1.82) a generalized signal $(q_1+\mu q_2)$ is used where μ is also an auxiliary parameter.

In matrix form the system (1.82) assumes the form

$$\dot{\vec{x}}=A\vec{x}+\vec{b}u,\ u=u(v),\ v=(\vec{g},\vec{x}), \tag{1.83}$$

where
$$\vec{x}=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix};\ A=\begin{bmatrix} -1 & -\beta_1 \\ -\alpha\beta_2 & -\alpha \end{bmatrix};\ \vec{b}=\begin{bmatrix} 1 \\ c \end{bmatrix};\ \vec{g}=\begin{bmatrix} 1 \\ \mu \end{bmatrix}.$$

The first problem which arises when investigating the problems of controlling the system (1.83) consists in the following. If the system (1.83) is in some initial state $\vec{x}_0$, does a continuous control input u(t) exist which will convert the system to the state $\vec{x}_1$ in the time t_1-t_0?

With respect to the systems of the type of (1.83) the answer to this question will depend on the properties of the matrix K:

$$K=[\vec{b},\ A\vec{b}]. \tag{1.84}$$

Namely, it is required that the rank of the matrix (1.84) be equal to two:

$$\text{rank } K = \text{rank } [\vec{b},\ A\vec{b}] = 2. \tag{1.85}$$

The matrix K for the given case has the form

$$K=\begin{bmatrix} 1 & -1-\beta_1 c \\ c & -\alpha\beta_2-\alpha c \end{bmatrix},$$

so that the condition (1.85) is equivalent to the condition det $K=c+\beta_1 c-\alpha\beta_2-\alpha c\neq 0$, that is, the condition of controllability of the initial system (1.82).

The equation

$$c+\beta_1 c-\alpha\beta_2-\alpha c=0 \tag{1.86}$$

defines the straight line c in the plane β_1, β_2.

Different cases of location of the straight lines (1.86) for different values of the parameters α, c are presented in Fig 1.17 (solid lines).

The next question which arises when investigating a controllable system (1.83) is the question of estimating the state of the system at the time t_0 by the known input and output effects measured in the future, that is, by the data of the functions u(t) and $\vec{x}(t)$ for $t \geq t_0$. This problem is connected with the fact that the required information is usually not completely available, and it must be reproduced by measuring the values which are accessible to the actual measurement process (in the given case, the values of $v=q_1+\mu q_2$).

If in the controllability criterion (1.85) an important role is played by the vector $\vec{b}'=[1, b]$, in the observability criterion, the vector $\vec{g}'=[1,\mu]$ plays an important role, and the corresponding result has the form:

the system (1.82) is observable if the rank of the matrix

$$G=[\vec{g}', A'\vec{g}'], \qquad (1.87)$$

where

$$G=\begin{bmatrix} 1 & -1-\alpha\mu\beta_2 \\ \mu & -\beta_1-\alpha\mu \end{bmatrix} \text{equals } 2. \qquad (1.88)$$

This condition is equivalent to the expression

$$\det G=\mu+\alpha\mu^2\beta_2-\beta_1-\alpha\mu\neq 0. \qquad (1.89)$$

The equation

$$\mu+\alpha\mu^2\beta_2-\beta_1-\alpha\mu=0 \qquad (1.90)$$

also corresponds to a straight line in the plane β_1, β_2. In Fig 1.17 the straight line (1.90) and (1.86) are depicted by dotted lines with the indexes ((μ), (c)).

Let us note the theoretical results of pertaining to the relation of the controllability and observability of the object. This relation is called the "duality" principle and in the given case consists in the following.

The system $\ddot{x}=A\vec{x}+\vec{b}u$, $u=u(v)$, $v=\vec{g}'\vec{x}$ is observable when and only when the system

$$\ddot{x}=A'\vec{x}+\vec{g}'u,\ u=u(\vec{v}),\ v=\vec{b}'\vec{x}$$

is controllable (the stroke indicates transposition of the matrices).

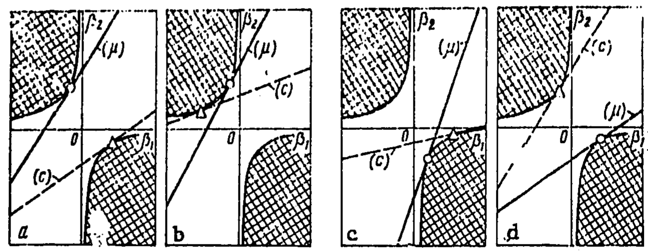


Figure 1.17. Boundaries of the regions of dynamic instability and lines of "uncontrollability" and "unobservability" for a double pendulum

There are also hyperbolas in Fig 1.17

$$\beta_1\beta_2 = -\frac{(1-\alpha)^2}{4\alpha}, \qquad (1.91)$$

which bound the regions of instability of an uncontrollable system (1.82) (for u=0), hereafter called the regions of natural instability (the double crosshatching).

Considering the systems of equations jointly

$$\begin{cases} \beta_1\beta_2 = -\frac{(1-\alpha)^2}{4\alpha}; \\ \alpha\beta_2 - c^2\beta_1 - c(1-\alpha) = 0 \ (c); \end{cases} \quad \begin{cases} \beta_1\beta_2 = -\frac{(1-\alpha)^2}{4\alpha}; \\ \alpha\mu^2\beta_2 + \mu(1-\alpha) - \beta_1(\mu), \end{cases} \qquad (1.92)$$

we note that tangency of the hyperbola (1.91) and straight lines (c) and (μ) occurs at the point

$$(-1-\alpha/2c;\ c(1-\alpha)/2\alpha);\ (\mu(1-\alpha)/c;\ -1-\alpha/2\alpha\mu),$$

shown in Fig 1.17.

Noting this fact, let us return to another important problem characteristic of linear stationary controlled systems -- the problem of control of the eigenvalues. As before, system (1.82) is used as the model.

Modal Equation of a Dynamically Unstable System

The term modal is used because the roots of the characteristic equation correspond to the components of free motion sometimes called modes.

The problem consists in proper selection of the function u as a function of the state of the system

$$u=-\vec{k}'\vec{x} \qquad (1.93)$$

so that the equation of the closed system

$$\dot{\vec{x}}=(A-\vec{b}\vec{k}')\vec{x}. \qquad (1.94)$$

will have a characteristic equation with previously prescribed arrangement of the roots.

Hence, it is possible to draw the conclusion that the condition of control of the eigenvalues of the closed system (1.83) is complete controllability of the object [the condition (1.85)].

Let us consider this problem as applied to the case where the uncontrollable ($u(t)\equiv0$) system is dynamically unstable. The purpose of the control then obviously is elimination of the harmful effect of the asymmetric cross relations which are the initial cause of the instability. It is possible, for example, to require that the roots of the system (1.94) be equal to the roots of the initial partial system (in the absence of the relation $\beta_1\beta_2$ and equation u(t)).

Thus, let us propose that the system (1.83) is controllable. Then the canonical form

$$\dot{\hat{\vec{x}}}=A\hat{\vec{x}}+\hat{\vec{b}}u, \qquad (1.95)$$

will exist, where

$$A=\begin{bmatrix}0 & 1\\ -\alpha+\alpha\beta_1\beta_2 & -(1+\alpha)\end{bmatrix},\quad \hat{\vec{b}}=\begin{bmatrix}0\\1\end{bmatrix}.$$

The matrix of transition from the representation $\{A, \vec{b}\}$ to the representation $\{A, \hat{\vec{b}}\}$ is unique and is calculated by the formula $P=[\hat{b}, A\hat{b}][b, Ab]^{-1}$.

Performing the required transformations, we find

$$P=\frac{1}{\tau_2-c\tau_1}\begin{bmatrix}-c & 1\\ -\alpha\beta_2+c & \beta_1c-\alpha\end{bmatrix},\quad P^{-1}=\begin{bmatrix}\alpha-\beta_1c & 1\\ c-\alpha\beta_2 & c\end{bmatrix},$$

where $\tau_1=-1-\beta_1c;\ \tau_2=-\alpha\beta_2-\alpha c.$

Let the characteristic equation $\phi_A=p^4+(1+\alpha)p^2+\alpha(1-\beta_1\beta_2)=0$ have the roots $p_{1,2}=-\delta\pm i\omega$; $p^2_{3,4}=\delta\pm i\omega$ (system (1.82) is unstable for $u(t)\equiv0$).

Let us find the control $u(t)=-\vec{k}'\vec{x}(t)$, where $\vec{k}'=[k_1, k_2]$ is the unknown vector such that the characteristic equation of the closed system has the given form:

$$\varphi_u = p^4 + (\Omega_1^2 + \Omega_2^2)p^2 + \Omega_1^2\Omega_2^2 = 0,$$

where $\Omega_1^2 \cdot \Omega_2^2$ are new (desirable) values of the squares of natural frequencies of the system.

The vector $\hat{k}'=[k_1, k_2]$ in the canonical system (1.95) is calculated most simply. Actually, substituting the values of $u(t)=-\hat{\vec{k}}'\vec{x}(t)$ in equation (1.95), it is possible to arrive at the characteristic equation

$$\begin{vmatrix} -p^2 & 1 \\ -\alpha + \alpha\beta_1\beta_2 - \hat{k}_1 & -(1+\alpha) - \hat{k}_2 - p^2 \end{vmatrix} = 0,$$

or in expanded form, the equation

$$p^4 + p^2(1 + \alpha + \hat{k}_2) + (\hat{k}_1 + \alpha - \alpha\beta_1\beta_2) = 0. \qquad (1.96)$$

Comparing the coefficients of the polynomials ϕ_A and ϕ_u we find

$$\begin{aligned} \hat{k}_1 &= \Omega_1^2\Omega_2^2 - (\alpha - \alpha\beta_1\beta_2); \\ \hat{k}_2 &= \Omega_1^2 + \Omega_2^2 - (1 + \alpha). \end{aligned} \qquad (1.97)$$

In order to find the coefficients k_1, k_2 for the initial system (1.94), it is necessary to use the formulas of transformation from the representation $\{A, \vec{b}\}$ to the representation $\{\hat{A}, \hat{\vec{b}}\}$:

$$\hat{\vec{x}} = P\vec{x};\ \hat{A} = PAP^{-1};\ \hat{\vec{b}} = P\vec{b};\ \vec{k}' = \hat{\vec{k}}'P. \qquad (1.98)$$

Using the latter equality, we find:

$$\begin{aligned} k_1 &= \frac{-c\left[\Omega_1^2\Omega_2^2 - (\alpha - \alpha\beta_1\beta_2)\right] + \left[\Omega_1^2 + \Omega_2^2 - (1+\alpha)\right](c - \alpha\beta_2)}{c + \beta c^2 - \alpha c - \alpha\beta_2}; \\ k_2 &= \frac{\left[\Omega_1^2\Omega_2^2 - (\alpha - \alpha\beta_1\beta_2)\right] + \left[\Omega_1^2 + \Omega_2^2 - (1+\alpha)\right](\beta_1 c - \alpha)}{c + \beta_1 c^2 - \alpha c - \alpha\beta_2}. \end{aligned} \qquad (1.99)$$

For example, let $\Omega_1^2 = 1$; $\Omega_2^2 = \alpha$.

In other words, let us require that the closed system (1.82) have the same roots as the open system ($u(t)\equiv 0$) in the absence of a cross relation $\alpha\beta_1\beta_2$; then, substituting the values of Ω_1^2, Ω_2^2 in formulas (1.99) we find

$$k_1 = -\alpha c \frac{\beta_1\beta_2}{\tau_2 - c\tau_1}; \quad k_2 = \frac{\alpha\beta_1\beta_2}{\tau_2 - c\tau_1}.$$

Let us note that in the general case the choice of k_1 and k_2 is not unique. Therefore, the possibility for optimization of control is retained (if the appropriate criterion is selected).

Considering the problem of control of the natural frequencies of the system (1.82), the control is represented as a factor which neutralizes the destabilizing effect of the asymmetric cross relation $\beta_1 \neq \alpha\beta_2$. Therefore the problem of selecting the coefficients k_1, k_2 can be stated as follows: it is necessary to find the values of k_1, k_2 for which the difference of the natural frequencies $\Omega_1{}^2, \Omega_2{}^2$ of the system assumes the given value for the additional condition

$$\Phi = k_1^2 + k_2^2 \Rightarrow \min. \qquad (1.100)$$

For example, let us require that in the characteristic equation of the closed system

$$\Omega_1^2 = \Omega^2; \; \Omega_2^2 = \Omega^2 + \Delta\Omega^2 \qquad (1.101)$$

the frequency difference $\Delta\Omega$ assumes the given value α^2.

Then

$$\begin{aligned} \hat{k}_1 &= \Omega^4 + \alpha^2\Omega - \alpha(1 - \tilde{\varkappa}); \\ \hat{k}_2 &= 2\Omega^2 + \alpha^2 - \alpha - 1. \end{aligned} \qquad (1.102)$$

Minimization of the sum of the squares $\hat{\Phi} = \hat{k}_1^2 + \hat{k}_2^2$ leads to the following cubic equation with respect to the unknown value of $\Omega^2 = \zeta$:

$$2\zeta^3 + (3\alpha^2 + 2)\zeta^2 + [-2\alpha(1 - \varkappa) + \alpha^4 + 4]\zeta + [-2 - 2\alpha(1 - \varkappa)] = 0.$$

For example, let us set $\alpha=2$; $\tilde{\chi}=-3$. We obtain the equation $\zeta^3+6\zeta^2+2\zeta-15=0$, having the positive solutions $\zeta=\Omega^2=1.303$.

Substituting the calculated value of Ω^2 and the values of the parameters α, $\tilde{\chi}$ in the formulas (1.102), we obtain a relative result.

The given difference $\Delta\Omega 2 = \Omega_2{}^2 - \Omega_1{}^2$ of the natural frequencies of the system (1.82) is insured for values of $\hat{k}_1 = -1.091$; $\hat{k}_2 = 3{,}606$, satisfying the criterion (1.100). Here $\Omega_1{}^2 = 1.303$; $\Omega_2{}^2 = 5.303$.

The investigated problem permits the following geometric interpretation. In the plane $\hat{k}_1$, $\hat{k}_2$ let us consider the family of circles

$$\hat{k}_1^2+\hat{k}_2^2=r^2 \qquad (1.103)$$

of variable radius r (Fig 1.18), for the frequencies Ω^2, $\Omega^2+\Delta\Omega^2$ satisfy the equation

$$p^4+(2\Omega^2+\Delta\Omega^2)p^2+(\Omega^2+\Delta\Omega)\Omega^2=0, \qquad (1.104)$$

and on the other hand, the characteristic equation of the closed system has the form

$$p^4+p^2(1+\alpha+\hat{k}_2)+(\hat{k}_1+\alpha-\alpha\beta_1\beta_2)=0, \qquad (1.105)$$

then, comparing (1.104) and (1.105), we find

$$\Delta\Omega^2=[1+\alpha+\hat{k}_2]^2-4[\alpha-\alpha\varkappa+k_1]=\text{const.} \qquad (1.106)$$

The equation (1.106) in the plane $\hat{k}_1$, $\hat{k}_2$ defines the parabola

$$\hat{k}_1=\frac{1}{4}[1+\alpha+\hat{k}_2]-\alpha(1-\varkappa)-\frac{1}{4}\Delta\Omega^2.$$

As is obvious from the Figure 1.18, the values found for $\hat{k}_1$* and $\hat{k}_2$* determine the minimum radius of the circle (1.103) for which tangency of the curves (1.103)-(1.106) is insured, that is, the condition (1.100) is satisfied.

In our example $r=\sqrt{\hat{k}_1^2+\hat{k}_2^2}=3{,}767.$

The problem of modal control of the system (1.82) as a whole also permits geometric interpretation. Actually, let us consider the characteristic equation of the system (1.96) and let us rewrite it in the form ($\lambda=-p^2=\mu+i\Omega$):

$$\lambda^2+(1+\alpha+\hat{k}_2)\lambda+\alpha(1-\varkappa)+\hat{k}_1=0. \qquad (1.107)$$

Let us apply the procedure of the root hodograph method to equation (1.107), constructing the biparametric family of trajectories of the roots (with respect to the parameters $\hat{k}_1$, $\hat{k}_2$).

Let us denote by $\Phi_2=\lambda^2+(1+\alpha+k_2)\lambda+\alpha(1-\chi)=0$ the equation for determining the initial points ρ_ν ($\nu=1,2$) of the root trajectories. On variation of the parameter k_2 ($0\leq k_2<\infty$) the initial points are located on the arc of a circle (the dotted line in Fig 1.19). Since there are no limiting points in the given case, for fixed values of $\hat{k}_2$ the trajectories of the roots (for $0\leq\hat{k}_1<\infty$) are straight lines parallel to the axis Ω (the dash-dot lines in Fig 1.19). The root trajectories are shown in the same figure also for negative values of the parameter $\hat{k}_1$ (the solid lines).

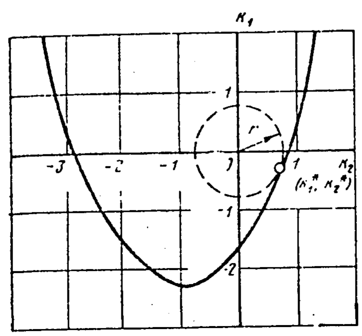


Figure 1.18 Calculation of the optimal parameters k_1^*, k_2^* of the control system ($k_1^*=-0.88$; $k_2^*=2.34$)

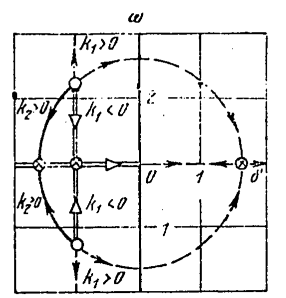


Figure 1.19 Root hodographs for a double pendulum: parameters -- coefficients, k_1, k_2

Hence, it is clear in what direction it is necessary to change the parameters $\hat{k}_1$ and $\hat{k}_2$ so that the boundaries of the region of instability will be reached as fast as possible: it is necessary to increase the parameter $\hat{k}_2$ and decrease the parameter $\hat{k}_1$ as much as possible.

For the given values of $\hat{k}_2$ it is easy to calculate the boundary value Re $\lambda=\mu^*=-(1+\alpha+\hat{k}_2)/2$.

Then the critical value of the parameter k_1 can be calculated from the expression $k_1^*=-\mu^{*2}+(1+\alpha+k_2)\mu^*+\alpha(1-\varkappa)$.

For our example, if $k_2^*=3.606$, then $k_1^*=-1.091$.

Special Cases

The condition of controllability (1.85), as is obvious, guarantees the possibility of modal control of the system. At the same time this condition is a sufficient condition; therefore, expression (1.86) still does not mean that the given control cannot change the structure of the system, for example, making it stable.

Let us consider in more detail what the physical meaning of the term "uncontrollability" is as applied to the problem of stabilization of a dynamically unstable system (1.82).

For the analysis it is necessary to again turn to the system of equations (1.82), for in the case of uncontrollability, the canonical system $\{\hat{A},\hat{b}\}$ does not exist.

Thus, let us consider the controllable oscillatory system of the type

$$\ddot{q}_1 + q_1 + \beta_1 q_2 = u; \qquad (1.108)$$
$$\ddot{q}_2 + \alpha q_2 + \alpha\beta_2 q_1 = cu;$$
$$u = -(k_1 q_1 + k_2 q_2). \qquad (1.109)$$

Let the system (1.82) as an object of control be uncontrollable, that is, let the condition (1.86) be satisfied:

$$c + \beta_1 c^2 + c - \alpha\beta_2 - \alpha c = 0. \qquad (1.110)$$

On the basis of expression (1.86)

$$\alpha\beta_2 = c^2\beta + c(1-\alpha). \qquad (1.111)$$

The characteristic equation of the closed system (1.108)-(1.109) considering the condition (1.111) will be obtained in the form

$$\begin{vmatrix} p^2 + 1 + k_1 & \beta_1 + k_2 \\ -\alpha c + c + \beta_1 c^2 + k_1 c & p^2 + \alpha + k_2 c \end{vmatrix} = 0,$$

or in expanded form:

$$p^4 + (1 + \alpha + Q)\, p^2 + (\alpha - c\beta_1)(1 + c\beta_1 + Q) = 0, \qquad (1.112)$$

where

$$Q = k_1 + c k_2.$$

It is easy to show that the discriminant of this equation

$$D = (1 + \alpha + Q)^2 - 4(\alpha - c\beta_1)(1 + c\beta_1 + Q)$$

can be represented in the form

$$D = [Q + (1 - \alpha + 2c\beta_1)]^2,$$

consequently, the roots of the characteristic equation have the following simple form:

$$p_1^2 = -\alpha + c\beta_1; \quad p_2^2 = -(1 + c\beta_1) - Q. \qquad (1.113)$$

In this paper the assumption has been made everywhere that the coefficients of the characteristic equation [including the equations (1.112)] are positive, which means satisfaction of the necessary conditions of stability of the corresponding systems. Then, as is easy to see, the roots (1.113) are real and negative [the system (1.108)-(1.109) is stable].

It is characteristic that one of the roots does not depend on the controlling parameter Q. This means that it is possible to choose only the root

$$p_2^2 = -(1+c\beta_1) - (k_1 + ck_2),$$

by appropriately selecting the values of k_1 and k_2. Obviously, this is the meaning of the term "uncontrollability" for the system (1.82).

However, the problem here is a different one. It consists in insuring stability of the system (1.82) by appropriate selection of the control, including in the case of uncontrollability, that is, when the system parameters α_1, β_1, β_2 and the control parameter c are related by the condition (1.111).

Obviously, the solution to the problem exists. It is convincing that the nature of the relation (1.111) is such that even for Q=0 the system is dynamically stable and it is possible to use the freedom in the selection of the parameter Q to insure the required interval between the new natural frequencies of the system

$$\omega_1^2 = \alpha - c\beta_1;\ \omega_2^2 = 1 + c\beta_1 + Q.$$

This fact is, of course, clear from the graphs in Fig 1.17. It is sufficient to note that the straight line (1.111) is entirely located in the region of dynamic stability of the system (1.82), with the exception of the point

$$\beta_1 = -\frac{1-\alpha}{2c};\ \beta_2 = \frac{c(1-\alpha)}{2\alpha}. \qquad (1.114)$$

In this case

$$p_1^2 = -\frac{1+\alpha}{2};\ p_2^2 = -\frac{1+\alpha}{2} - Q.$$

Here the value of $Q=k_1+ck_2$ is equal to the interval $\Delta\omega^2$ between the natural frequencies of the controlled system (1.108)-(1.109).

Thus, it is possible to insure dynamic stability of the system (1.82) both in the case of controllability and in the case of uncontrollability of the system [in the sense of the criterion (1.85)]. In the last case, although it is impossible to assign given values to the roots of the closed system by selecting k_1 and k_2 it is possible to insure a "interval of safety" between the natural frequencies, at the same time, compensating for the unfavorable effect of the asymmetric cross relation $\beta_1\beta_2$.

The unconservative oscillatory system with two degrees of freedom is a convenient model for demonstration of an entire series of interesting dynamic effects connected with its potential instability (flutter, shimmy [31], and so on).

At the same time, such systems are of interest also in connection with the modern control problems. Here, as is obvious from the presented example, classical instability appears as one of the properties (jointly with controllability and observability) of the system as an object of control. These properties, being closely related, on the whole quite completely determine the dynamic portrait of the investigated system.

Let us note that in the literature the problems of controllability and observability are discussed as a necessary element of the problem of optimal control. In this book we are interested in the mentioned properties in connection with the theory of stabilizability of liquid-propellant rockets interpreted as oscillatory objects with a large number of degrees of freedom, under conditions of incompleteness of the information used for control.

Here the first problem which arises is not the problem of optimal control, but the problem of whether it is possible in general to control the entire system of oscillatory elements for some given structure of the control system (the automatic stabilization system) and in what cases amplitude or phase stabilization of the object is expedient.

The investigated examples where the physical meaning of the concepts of uncontrollability, unobservability and the possibility of modal control of the system connected with them and also the specific role of the dissipative forces are discussed, illustrate the class of problems of the dynamics of the active segment of the space vehicle which is the subject of the investigation in the following chapters.

CHAPTER 2. STABILIZABILITY OF OSCILLATORY SYSTEMS WITH ONE INPUT

2.1. Terminology: Stabilizability, Structural Properties, Stability. Formulation of the Basic Problems

Two Problems of Motion Control Theory

Let us consider the problem of selecting the parameters of a linear control system in the form proposed by Yu. M. Berezanskiy. Let the equations of motion and the initial conditions for some system have the form

$$\dot{\vec{x}}=\tilde{A}\vec{x},\ \vec{x}(0)=\vec{x}_0, \qquad (2.1)$$

where $\vec{x}=(x_1, x_2, \ldots, u)$, u(t) is the controlling function;

$$\tilde{A}=\left\|\begin{matrix} a_{11}a_{12}\ldots & a_{1n}b_1 \\ a_{21}a_{22}\ldots & a_{2n}b_2 \\ \ldots\ldots & \ldots \\ a_{n1}a_{n2}\ldots & a_{nn}b_n \\ \varkappa_1\ \ \varkappa_2\ldots & \varkappa_n\varkappa \end{matrix}\right\|. \qquad (2.2)$$

Let the following be given: a) the elements of the matrix $A=\|a_{ij}\|_1^n$; b) the vector $\vec{b}=\{b_1,\ldots,b_n\}$ (or the vector $\vec{\varkappa}=\{\varkappa_1,\ldots,\varkappa_n\}$.

It is necessary to find the vector $\varkappa$ (the vector $\vec{b}$ respectively) such that the matrix A will have eigenvalues given in advance

$$\mu_1, \mu_2, \ldots, \mu_n. \qquad (2.3)$$

Let us write the characteristic equation of the matrix (2.2) in the form

$$\lambda+\varkappa-\sum_{n=1}^{n}\frac{p_r p_r^*}{\lambda-\lambda_r}=0, \qquad (2.4)$$

where

$$p_r = \sum_{k=1}^{n} \varkappa_k \varphi_k^r; \quad p_r^* = \sum_{j=1}^{n} b_j \varphi_j^{*r}; \tag{2.5}$$

$\phi^r (\phi^*)^r$ is the eigenvector of the matrix A ($\tilde{A}$) corresponding to the eigenvalue $\lambda_r (\tilde{\lambda}_r)$. Let us set

$$p_r p_r^* = -\beta_r. \tag{2.6}$$

The equation (2.4) assumes the form

$$\lambda + \varkappa + \sum_{r=1}^{n} \frac{\beta_r}{\lambda - \lambda_r} = 0. \tag{2.7}$$

Let us show in advance that by choosing the edge elements b_j, $\varkappa_k$, $\varkappa$ in the matrix $\tilde{A}$ it is possible to achieve the situation where the eigenvalues of the bordered matrix will assume any values given in advance or, what amounts to the same thing, the coefficients in the characteristic equation of the matrix $\tilde{A}$ can be made whatever one might like.

Let us denote by $z_1, z_2, \ldots, z_m$ the roots of the polynomial

$$P(z, m, z_1, z_2, \ldots, z_m) = (z - z_1)(z - z_2)\ldots(z - z_m) = \\ = z^m + \sigma_1(m, z_1, \ldots, z_m) z^{m-1} + \ldots + \sigma_m(m, z_1, \ldots, z_m), \tag{2.8}$$

the coefficients of which σ_j are elementary symmetric functions satisfying the following expressions:

$$\begin{aligned} \sigma_1(m, z_1, \ldots, z_m) &= -(z_1 + \ldots + z_m) = \\ &= \frac{1}{(m-1)!} P^{m-1}(0, m, z_1, \ldots, z_m); \\ \sigma_2(m, z_1, \ldots, z_m) &= z_1 z_2 + z_1 z_3 + \ldots + z_{m-1} z_m = \\ &= \frac{1}{(m-2)!} P^{m-2}(0, m, z_1, \ldots, z_m); \\ \sigma_3(m, z_1, \ldots, z_m) &= -(z_1 z_2 z_3 + z_1 z_2 z_4 + \ldots) = \\ &= \frac{1}{(m-3)!} P^{m-3}(0, m_1 z_1, \ldots, z_m); \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \sigma_m(m, z_1, \ldots, z_m) &= (-1)^m z_1 z_2 z_3, \ldots, z_m = P(0, m_1, z_1, \ldots, z_m). \end{aligned} \tag{2.9}$$

From (2.7) we find

$$(\lambda + \varkappa) \prod_{l=1}^{n} (\lambda - \lambda_l) + \sum_{j=1}^{n} b_j \prod_{l \neq j}^{n} (\lambda - \lambda_l) = 0. \tag{2.10}$$

Considering the introduced notation let us represent the lefthand side of the equation (2.10) in the form of a polynomial with respect to powers of λ:

$$\lambda^{n+1}+[\varkappa+\sigma_1(n,\ \lambda_1,\ldots,\ \lambda_n)]\lambda^n+\Big[\varkappa\sigma_1(n,\ \lambda_1,\ldots,\ \lambda_n)+\sigma_2(n,\ \lambda_1,\ldots,\ \lambda_n)+$$
$$+\sum_{j=1}^{n}b_j\Big]\lambda^{n-1}+[\varkappa\sigma_2(n,\ \lambda_1,\ldots,\ \lambda_n)+\sigma_3(n,\ \lambda_1,\ldots,\ \lambda_n)+b_1\sigma_1\times$$
$$\times(n-1,\ \lambda_2,\ldots,\ \lambda_n)+\ldots+b_n\sigma_1(n-1,\ \lambda_1,\ldots,\ \lambda_{n-1}]\lambda^{n-2}+$$
$$+[\varkappa\sigma_3(n,\ \lambda_1,\ldots,\ \lambda_n)+\sigma_4(n,\ \lambda_1,\ldots,\ \lambda_n)+b_1\sigma_2(n-1,\ \lambda_2,\ldots,\ \lambda_n)+\ldots+$$
$$\ldots+b_n\sigma_2(n-2,\ \lambda_1,\ldots,\ \lambda_{n-1}]\lambda^{n-3}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+[\varkappa\sigma_n(n,\ \lambda_1,\ldots,\ \lambda_n)+\sigma_n(n,\ \lambda_1,\ldots,\ \lambda_n)+b_1\sigma_{n-1}(n-1,\ \lambda_2,\ldots,\ \lambda_n)+\ldots$$
$$\ldots+b_n\sigma_{n-1}(n-1,\ \lambda_1,\ldots,\ \lambda_{n-1})]=0. \qquad (2.11)$$

Let $c_1, c_2, \ldots, c_{n+1}$ be arbitrarily given numbers. Let us demonstrate that it is always possible to select the parameters χ, b_j such that the coefficients of the equation (2.11) will assume the values $c_1, c_2, \ldots, c_{n+1}$. From (2.11) it follows that the unknowns χ, b_j must satisfy the following system of linear nonuniform equations:

$$\varkappa+\sigma_1(n,\ \lambda_1,\ldots,\ \lambda_n)=c_1;$$
$$\varkappa\sigma_2(n,\ \lambda_1,\ldots,\ \lambda_n)+\sigma_3(n,\ \lambda_1,\ldots,\ \lambda_n)+b_1\sigma_1(n-1,\ \lambda_2,\ldots,\ \lambda_n)+\ldots$$
$$\ldots+b_n\sigma_1(n-1,\ \lambda_1,\ldots,\ \lambda_{n-1})=c_2;$$
$$\varkappa\sigma_3(n,\ \lambda_1,\ldots,\ \lambda_n)+\sigma_4(n,\ \lambda_1,\ldots,\ \lambda_n)+b_1\sigma_2(n-1,\ \lambda_2,\ldots,\ \lambda_n)+\ldots$$
$$\ldots+b_n\sigma_2(n-1,\ \lambda_1,\ldots,\ \lambda_{n-1})=c_3; \qquad (2.12)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\varkappa\sigma_n(n,\ \lambda_1,\ldots,\ \lambda_n)+b_1\sigma_{n-1}(n-1,\ \lambda_2,\ldots,\ \lambda_n)+\ldots$$
$$\ldots+b_n\sigma_{n-1}(n-1,\ \lambda_1,\ldots,\ \lambda_{n-1})=c_{n+1}.$$

The determinant of this system has the form:

$$\Delta_\sigma=\begin{vmatrix} 1 & 0 & \ldots & 0 \\ \sigma_1(n,\ \lambda_1,\ldots,\ \lambda_n) & 1 & \ldots & 1 \\ \sigma_2(n,\ \lambda_1,\ldots,\ \lambda_n) & \sigma_1(n-1,\ \lambda_2,\ldots,\ \lambda_n) & \ldots & \sigma_1(n-1,\ \lambda_1,\ldots,\ \lambda_{n-1}) \\ \sigma_3(n,\ \lambda_1,\ldots,\ \lambda_n) & \sigma_2(n-1,\ \lambda_2,\ldots,\ \lambda_n) & \ldots & \sigma_2(n-1,\ \lambda_1,\ldots,\ \lambda_{n-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n-1}(n,\ \lambda_1,\ldots,\ \lambda_n) & \sigma_{n-2}(n-1,\ \lambda_2,\ldots,\ \lambda_n) & & \sigma_{n-2}(n-1,\ \lambda_1,\ldots,\ \lambda_{n-1}) \\ \sigma_n(n,\ \lambda_1,\ldots,\ \lambda_n) & \sigma_{n-1}(n-1,\ \lambda_2,\ldots,\ \lambda_n) & & \sigma_{n-1}(n-1,\ \lambda_1,\ldots,\ \lambda_{n-1}) \end{vmatrix}. \qquad (2.13)$$

In order to establish the resolvability of the system (2.12) for any righthand sides of the type $c_1-\sigma_1(n, \lambda_1,\ldots,\lambda_n),\ldots, c_n-\sigma_n(n, \lambda_1,\ldots,\lambda_n)$, c_{n+1}, it is necessary to demonstrate the determinant Δ_σ is nonzero.

Expanding this determinant with respect to elements of the first row and multiplying the first row of the determinant obtained by (n-1)!, the second by (n-2)! and so on, we find:

$$\bar{\Delta}_\sigma=\begin{vmatrix} P^{n-1}(0, n-1, \lambda_2,\ldots,\lambda_n) & P^{n-1}(0, n-1, \lambda_1, \lambda_3,\ldots,\lambda_n)\ldots & \ldots P^{n-1}(0, n-1, \lambda_1,\ldots,\lambda_{n-1}) \\ P^{n-2}(0, n-1, \lambda_2,\ldots,\lambda_n) & P^{n-2}(0, n-1, \lambda_1, \lambda_3,\ldots,\lambda_n)\ldots & \ldots P^{n-2}(0, n-1, \lambda_1,\ldots,\lambda_{n-1}) \\ P(0, n-1, \lambda_2,\ldots,\lambda_n) & P(0, n-1, \lambda_1, \lambda_3,\ldots,\lambda_n)\ldots & \ldots P(0, n-1, \lambda_1,\ldots,\lambda_n) \end{vmatrix}. \quad (2.14)$$

$$\bar{\Delta}_\sigma=(n-1)!(n-2)!\ldots 1!\Delta_\sigma.$$

The determinant (2.14) coincides with the value at zero of the Wronskian for the system of functions:

$$P(\lambda, n-1, \lambda_2, \lambda_3,\ldots,\lambda_n), P(\lambda, n-1, \lambda_1, \lambda_3,\ldots,\lambda_n),\ldots \ldots, P(\lambda, n-1, \lambda_1,\ldots,\lambda_{n-1}). \quad (2.15)$$

These functions are linearly independent. Actually, let us write the linear combination of functions (2.15):

$$\alpha_1 P(\lambda, n-1, \lambda_2, \lambda_3,\ldots,\lambda_n)+\alpha_2 P(\lambda, n-1, \lambda_1, \lambda_3,\ldots,\lambda_n)+\ldots \ldots+\alpha_n P(\lambda, n-1, \lambda_1,\ldots,\lambda_{n-1})=0. \quad (2.16)$$

Substituting in expression (2.16) $\lambda=\lambda_1$, we find

$$\alpha_1 P(\lambda_1, n-1, \lambda_2, \lambda_3,\ldots,\lambda_n)=0,$$

for $P(\lambda_1, n-1, \lambda_2, \lambda_3,\ldots,\lambda_n)\neq 0$, then $\alpha_1=0$.

Analogously, substituting $\lambda=\lambda_2$, $\lambda=\lambda_3,\ldots$, we find $\alpha_2=0$, $\alpha_3=0,\ldots, \alpha_n=0$.

Thus the equality (2.16) is possible only if $\alpha_j=0$ ($j=1, 2,\ldots, n$) which indicates the linear independence of the functions (2.15).

On the other hand, the functions (2.15) as polynomials of degree (n-1) satisfy the equation

$$\frac{d^n P}{d\lambda^n}=0. \quad (2.17)$$

The linear independence of the solutions to equation (2.17) indicates vanishing of the Wronskian Δ_σ. Consequently, the system (2.12) is resolvable, Q.E.D.

Thus, being given the values of $\mu_1, \mu_2, \ldots, \mu_{n+1}$ (the eigenvalues of the matrix A) for finding the unknown χ, χ_i, b_i ($i=1, 2, \ldots, n$), it is necessary to proceed as follows:

Calculate the coefficients of the polynomial c_j ($j=1,2,\ldots n+1$);

Calculate the functions σ_k ($k=1,2,\ldots,m$);

From equations (2.12) find χ, $b_1, \ldots, b_n$;

Calculate the values of

$$p_r p_r^* = -\beta_r (r=1,\ 2, \ldots,\ n); \qquad (2.18)$$

Being given p_r, p_r^* arbitrarily so that the condition (2.18) will be satisfied, solve the system:

$$p_r = \sum_{k=1}^{n} \chi_k \varphi_k^2$$

with respect to χ_k.

First let the column $\vec{b}=\{b_1, \ldots, b_n\}$ be given. It is necessary to select the edge row $\vec{\chi}=\{\chi_1, \ldots, \chi_n\}$.

Obviously, the problem is resolvable when and only when all of the numbers p_r^* calculated from the relation (2.5) are nonzero according to the given values of b_j.

In this case the vector $\vec{\chi}$ is expressed linearly in terms of the vector $\vec{p}=\{p_1, \ldots, p_n\}$:

$$p_j = -\frac{\beta_j}{p_j^*},$$

which means, in terms of the vector $\vec{b}$.

All of the numbers p_r^* are nonzero, if the scalar products

$$(\vec{a},\ \vec{\varphi}^{*r}) \neq 0 \ (r=1,\ 2, \ldots,\ n). \qquad (2.19)$$

Usually the unknown eigenvectors $\vec{\phi}^*$ of the matrix A* figure in this condition; therefore it is difficult to check (2.19). Let us obtain the resolvability criterion of the given problem in a different form.

Let us expand the vector $\vec{b}$ with respect to the eigenvectors $\vec{\phi}^1, \vec{\phi}^2, \ldots, \vec{\phi}^{n-1}, \vec{\phi}^n$ of the matrix A and let us apply the matrix A to this expansion (n-1) times.

We obtain:

$$\begin{aligned}
\vec{b} &= (\vec{b}, \vec{\varphi}^{*1})\vec{\varphi}^1 + (\vec{b}, \vec{\varphi}^{*2})\vec{\varphi}^2 + \ldots + (\vec{b}, \vec{\varphi}^{*n})\vec{\varphi}^n; \\
A\vec{b} &= \lambda_1(\vec{b}, \vec{\varphi}^{*1})\vec{\varphi}^1 + \lambda_2(\vec{b}, \vec{\varphi}^{*2})\vec{\varphi}^2 + \ldots + \lambda_n(\vec{b}, \vec{\varphi}^{*n})\vec{\varphi}^n; \\
A^2\vec{b} &= \lambda_1^2(\vec{b}, \vec{\varphi}^{*1})\vec{\varphi}^1 + \lambda_2^2(\vec{b}, \vec{\varphi}^{*2})\vec{\varphi}^2 + \ldots + \lambda_n^2(\vec{b}, \vec{\varphi}^{*n})\vec{\varphi}^n; \qquad (2.20) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
A^{n-1}\vec{b} &= \lambda_1^{n-1}(\vec{b}, \vec{\varphi}^{*1})\vec{\varphi}^1 + \lambda_2^{n-1}(\vec{b}, \vec{\varphi}^{*2})\vec{\varphi}^2 + \ldots + \lambda_n^{n-1}(\vec{b}, \vec{\varphi}^{*n})\vec{\varphi}^n.
\end{aligned}$$

Since the vectors $\vec{\varphi}^1, \vec{\varphi}^2, \ldots, \vec{\varphi}^n$ are linearly independent, assuming condition (2.19) to be satisfied, we conclude that the vectors $((\vec{b}\vec{\varphi}^{*1})\vec{\varphi}^1, \ldots, (\vec{b}, \vec{\varphi}^{*n})\vec{\varphi}^n)$ are linearly independent. According to (2.20) the vectors $\vec{b}, A\vec{b}, \ldots, A^{n-1}\vec{b}$ are obtained from the latter using the linear transformation with the matrix

$$U = \begin{vmatrix} 1 & 1 \ldots & 1 \\ \lambda_1 & \lambda_2 \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 \ldots & \lambda_n^2 \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_n^{n-1} \end{vmatrix}, \qquad (2.21)$$

the determinant of which is the Vandermondian, nonzero for $\lambda_i \neq \lambda_j$ $(i \neq j)$. Thus, the transformation with the matrix (2.21) is nonsingular, and therefore the vectors

$$\vec{b},\ A\vec{b},\ A^2\vec{b}, \ldots,\ A^{n-1}\vec{b}$$

are linearly independently.

In other words, the formulated problem (for the given vector $\vec{b}$) is resolvable when and only when the rank of the matrix

$$K = \|\vec{b},\ A\vec{b}, \ldots,\ A^{n-1}\vec{b}\|$$

is equal to n:

$$\Delta_k = |\vec{b},\ A\vec{b}, \ldots,\ A^{n-1}\vec{b}| = 0. \qquad (2.22)$$

Let us invert the problem, that is, let us assume that the vector $\vec{\chi}$ is given, and that it is necessary to find the vector $\vec{b}$. The condition of resolvability in the given case has the form

$$(\vec{g}, \vec{\varphi}^i) \neq 0 \ (i = 1, 2, \ldots, n). \qquad (2.23)$$

The systems of vectors $\{\phi^i\}$ and $\{\bar{\phi}^{*i}\}$ form a conjugate pair $(\varphi^i, \bar{\varphi}^{*j}) = \delta_{ij}$.

Here $A'\vec{\varphi}^{*j} = \overline{\lambda_j \varphi}^{*j}$.

We can therefore write:

$$\vec{g}=\sum_{i=1}^{n}(\vec{g},\vec{\varphi}^{i})\vec{\varphi}^{i};\ A'\vec{g}=\sum_{i=1}^{n}\lambda_i(\vec{g},\vec{\varphi}^{i})\vec{\varphi}^{*i};\ldots;(A')^{n-1}\vec{g}=$$
$$=\sum_{i=1}^{n}\lambda_i^{n-1}(\vec{g},\vec{\varphi}^{i})\vec{\varphi}^{*i}.$$

We have three system vectors:

$$\{\vec{\varphi}^{*i}\};\ \{(\vec{g},\vec{\varphi}^{i})\vec{\varphi}^{*i}\};\ldots;\ \{\vec{g}',\ A'\vec{g}',\ldots,(A)'^{n-1}\vec{g}'\}.$$

Their linear shells coincide; consequently, the linear independence of one of them implies linear independence of the rest. Thus, we arrive at the following condition of resolvability of the inverse problem (finding the vector $\vec{b}$): the rank of the matrix

$$G=\|\vec{g}',\ A'\vec{g}',\ldots,(A')^{n-1}\vec{g}'\|$$

is equal to n, or

$$\Delta_G=|\vec{g}',\ A'\vec{g}',\ldots,(A')^{n-1}\vec{g}'|\neq 0. \qquad (2.24)$$

The conditions (2.22), (2.24) in modern terminology are called the controllability and observability criteria of a linear controllable system. It is significant that they reflect not only the properties of the object (the properties of the matrix A), but also the properties of the object with respect to some correction device -- the properties of the pairs of matrices $(A, \vec{b})$, $(A, \vec{g})$.

Let us note that the modern view of the properties of controllability and observability of a linear object of control consists in the following.

The dynamic system

$$\dot{\vec{x}}(t)=A\vec{x}(t)+\vec{b}u(t),\ v=(\vec{g},\vec{x}) \qquad (2.25)$$

is called completely controllable if for each initial state $\vec{x}_0=\vec{x}(0)$ the piecewise continuous control u(t), t≥0 and the time $t_1<\infty$ are found such that the trajectory $\vec{x}(t)$ of the system corresponding to the control u(t) and the initial state $\vec{x}_0$, satisfies the condition $\vec{x}(t)=0$.

Thus, the controllability can be considered as a modern development of the constant of stability.

Then, the system (2.25) is said to be entirely observable with respect to the output

$$v=(\vec{g},\vec{x}),\ t_0\leqslant t\leqslant t_1,$$

if for any vector p the value of $(\vec{p}\vec{x}_0)$ at each point $\vec{x}_0$ can be restored by measuring v(t), $t_0 \leq t \leq t_1$.

The controllability and observability as structural properties of a linear object are related to each other by the principle of duality which in the general case consists in the following.

The linear stationary system

$$\dot{\vec{x}}(t) = A\vec{x}(t) + Bu(t), \quad \vec{y}(t) = C\vec{x}(t) \tag{2.26}$$

is observable when and only when the system

$$\dot{\vec{X}}(t) = A'X(t) + C'u(t), \quad \vec{y}(t) = B'\vec{X}(t) \tag{2.27}$$

is controllable.

Here A, B, C are matrices of the dimensions nXn, nXm, pXn, respectively. The vector u(t) has dimensionality mX1 [the system (2.26)] or pX1 [the system (2.27)].

Let us return to the other problem where the concept of structural properties is interpreted differently.

Let us consider an oscillatory system with one input

$$\dot{\vec{x}} = A\vec{x} + bu; \quad v = (\vec{g}, \vec{x}); \tag{2.28}$$

$$u = k_0 v + k_1 \dot{v}, \tag{2.29}$$

where the matrices A, $\vec{b}'$, $\vec{g}'$ have the following special form:

$$A = \begin{Vmatrix} 0 & a & a_{11} & a_{12} & \dots & a_{1n} \\ 0 & 0 & a_{21} & a_{22} & \dots & a_{2n} \\ 0 & 0 & \sigma_1^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \sigma_2^2 & \dots & 0 \\ \dots & & & & & \dots \\ \dots & & & & & \sigma_n^2 \end{Vmatrix};$$

$$\vec{b}' = \begin{Vmatrix} b_1 \\ b_2 \\ b_{\sigma 1} \\ b_{\sigma 2} \\ \vdots \\ b_{\sigma n} \end{Vmatrix}; \quad \vec{g}' = \begin{Vmatrix} 0 \\ 1 \\ -\eta_1' \\ -\eta_2' \\ \vdots \\ -\eta_m' \end{Vmatrix}.$$

The system (2.28)-(2.29) is characterized by the fact that in it control is realized only with respect to one parameter v, by which the controlling parameters k_0, k_1 are adjusted. Here the remaining oscillators (the auxiliary oscillators), generally speaking, can also be stable. The problem consists in investigating the stability of this system as the set of (n+2) oscillatory elements.

Let us first analyze the controllability and observability of the object of control (2.28), using the representations of (2.22), (2.24), for the matrices K and G. Calculating the mentioned matrices directly and excluding the generalized coordinate x_1 from the investigation, we obtain the following result.

The system (2.28) is controllable (for $\sigma_i \neq \sigma_j$), if

$$b_2 \neq 0;\ b_{\sigma_j} \neq 0\,(j=1,\ 2,\dots,\ m), \tag{2.30}$$

and observable if

$$\eta_j' \neq 0\ \ (j=1,\ 2,\dots,\ m). \tag{2.31}$$

The system (2.28) is controllable and observable if

$$b_{\sigma_j}\eta_j' \neq 0\ \ (j=1,\ 2,\dots,\ m). \tag{2.32}$$

As is obvious, the structural properties of the system are defined by the vectors b and g, the components of which $b_{\sigma j}$, η'_j play the role of characteristic system parameters.

The system of equations (2.28)-(2.29) is a linear system which usually is investigated with the application of well-developed frequency and operating methods. Let us use the approach based on the application of the Hermite-Bealer theorem.

The characteristic equation of the system (2.28)-(2.29), omitting the zero roots, is represented in the form

$$\Phi(p) = \Phi_0(p^2) + L(p)\,\Phi_k(p^2), \tag{2.33}$$

where

$$\Phi_0(p^2) = p^2 \prod_{j=1}^{m} (p^2 + \sigma_j^2);$$

$$\Phi_k(p^2) = -b_2 \left\{ \prod_{j=1}^{m} (p^2 + \sigma_j^2) - \sum_{j=1}^{m} c_j \eta_j' \prod_{i \neq j}^{m} (p^2 + \sigma_i^2) \right\}, \quad c_j = \frac{b_{\sigma_j}}{b_2}.$$

Let us transform the equation (2.33):

$$p^2\prod_{j=1}^{m}(p^2+\sigma_j^2)+\varkappa_0\left\{\prod_{j=1}^{m}(p^2+\sigma_j^2)-\sum_{j=1}^{m}c_j\eta_j'p^2\prod_{j=1}^{m}(p^2+\sigma_i^2)\right\}+$$
$$+\varkappa_1 p\left\{\prod_{j=1}^{m}(p^2+\sigma_j^2)-\sum_{j=1}^{m}c_j\eta_j'p^2\prod_{j=1}^{m}(p^2+\sigma_i^2)\right\}=0,$$

where $\varkappa_0=k_0b_2;\ \varkappa_1=k_1b_2.$

Let us use the following corollary of the Hermite-Bealer theorem.

In order that the polynomial $\Phi(p)=h(p^2)+pg(p^2$ be a Hurwitz polynomial, it is necessary and sufficient that the polynomials h(u), g(u), where $u=p^2$, make up a positive pair. In other words, the roots of the polynomials h(u), g(u), $u_1, u_2, \ldots, u_m$; $v_1, v_2, \ldots, v_m$ must be permutated in the following way respectively:

$$u_1<v_1<u_2<v_2<\ldots<v_{m-1}<u_m. \qquad (2.34)$$

In the given case

$$h(u)=h_0(u)+\varkappa_0 h_1(u);\quad g(u)=\varkappa_1 g_1(u);$$
$$h_0=u\prod_{j=1}^{n}(u+\sigma_j^2);\ h_1(u)=g_1(u)=\prod_{j=1}^{n}(u+\sigma_j^2)-\sum_{j=1}^{n}c_j\eta_j'u\prod_{j=1}^{m}(u+\sigma_i^2).$$

Let us simplify the analysis by proving the following statement. The polynomial

$$f(p)=[h_0(p^2)+\varkappa_0 h_1(p^2)]+\varkappa_1 p h_1(p^2) \qquad (2.35)$$

is a Hurwitz polynomial when and only when (for $\chi_0>0$) the polymomial

$$f_1(p)=h_0(p^2)+\varkappa_0 p h_1(p^2). \qquad (2.36)$$

has this property.

For the proof let us set:

$$h_0(p^2)=a_0p^{2m}+a_1p^{2(m-1)}+\ldots+a_{m-1}p^2;$$
$$h_1(p^2)=g_1(p^2)=b_0p^{2(m-1)}+b_1p^{2(m-2)}+\ldots+b_{m-1}.$$

For the polynomial (2.35) let us construct the Hurwitz matrix

$$\pi_\Gamma = \begin{Vmatrix} \varkappa_1 b_0 & \varkappa_1 b_0 & \varkappa_1 b_2 & \varkappa_1 b_3 \ldots 0 \\ a_0 & a_1 + \varkappa_0 b_0 & a_2 + \varkappa_0 b_1 & a_3 + \varkappa_0 b_2 \ldots 0 \\ 0 & \varkappa_1 b_0 & \varkappa_1 b_1 & \varkappa_1 b_2 \ldots 0 \\ 0 & a_0 & a_1 + \varkappa_0 b_1 & a_2 + \varkappa_0 b_1 \ldots 0 \\ 0 & 0 & \varkappa_1 b_0 & \varkappa_1 b_1 \ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 \ldots 0 \end{Vmatrix}. \qquad (2.37)$$

Subtracting the third row multiplied by $-\chi_0/\chi_1$ from the second row of the matrix (2.37), the fifth row multiplied by the same number from the fourth row, and so on, we obtain the matrix $\tilde{\pi}_\Gamma$ equivalent to π_Γ:

$$\tilde{\pi}_\Gamma = \begin{Vmatrix} \varkappa_1 b_0 & \varkappa_1 b_1 & \varkappa_1 b_2 & \varkappa_1 b_3 \ldots 0 \\ a_0 & a_1 & a_2 & a_3 \ldots 0 \\ 0 & \varkappa_1 b_0 & \varkappa_1 b_2 & \varkappa_1 b_3 \ldots 0 \\ 0 & a_0 & a_1 & a_2 \ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 \ldots \varkappa_0 b_{m-1} \end{Vmatrix}. \qquad (2.38)$$

The minors of the matrices π_Γ and $\tilde{\pi}_\Gamma$ (the Hurwitz determinants) for the polynomials (2.35)-(2.36) coincide, from which it follows that instead of the equation (2.35) with the matrix π_Γ it is possible to consider the equation (2.36) with the matrix (2.38), which proves the statement.

Thus, in accordance with the conditions of the formulated theorem for stability of the system (2.28)-(2.29) it is necessary that the roots of the equations be permutated

$$h_1(p^2) = p^2 \prod (p^2 + \sigma_i^2) = 0;$$

$$g_1(p^2) = \varkappa \left\{ \prod_{j=1}^{m} (p^2 + \sigma_j^2) - \sum_{j=1}^{m} c_j \eta_j' \prod_{j=1}^{m} (p^2 + \sigma_i^2) \right\} = 0$$

in the order indicated by the inequalities (2.34). For this purpose, obviously it is necessary that:

$$g_1(0) > 0; \quad g_1(\sigma_1^2) < 0; \quad g_1(\sigma_2^2) > 0; \ldots \qquad (2.39)$$

Substituting the values of $u=0$, $u=-\sigma_1^2; \ldots;$ $u=-\sigma_m^2$ in the function $g_1(u)$, successively, we find that the inequalities (2.39), and together with them, the conditions of stability of the given system (2.34) are satisfied if:

a) $\varkappa_0 > 0; \quad \varkappa_1 > 0$ (2.40) (the requirements on the parameters of the control system);

b) $$b_{\sigma j}\eta_j' > 0; \quad j=1, 2, \dots, m \qquad (2.41)$$

(the requirements on the parameters of the object of control).

As is obvious, the situation is such that the stability of the given system (2.28)-(2.29) can be insured as a result of the successive performance of two operations:

Adjustment of the control system parameters [in order that the conditions (2.40) be satisfied];

Selection of the parameters of the object of control in accordance with the inequalities (2.41) (having the sense of conditions of structural stability of the object).

Let us introduce the space of the parameters of the object of control of dimensionality 2m($2m(b_{\sigma 1}, \dots, b_{\sigma m}, \eta_1', \dots, \eta_m')$, in which the conditions of uncontrollability or observability of the system

$$b_{\sigma j}\eta_j' = 0 \quad (j=1, 2, \dots, m) \qquad (2.42)$$

are isolated by certain boundaries.

The conditions of structural stability (2.41) of the object of control provide the decoding of the regions on both sides of the boundary (2.42), which permits investigation of them as a generalization of the Kalman conditions as applied to the investigated special problem.

The establishment of the conditions of the type of (2.41) in the general case of oscillatory systems including n oscillators and constituting the object of control with one input is the basic problem of further analysis.

Here the central event is the theoretical possibility of the separation of investigation of the object of control from analysis of the closed system as a whole within the framework of reasonable assumptions, which in the given case leads to the necessity for introducing the concept of controllability and observability of the system (problem 1), structural stability (problem 2), and then the concept of stabilizability of the object of control.

Formalization of the Object of Control and the Control System. Statement of the Problem

Let us consider the system of differential equations

$$\ddot{\vec{x}} = \varepsilon B\dot{\vec{x}} + A\vec{x} + \vec{b}u, \qquad (2.43)$$

where $\vec{x}$ is the vector of the generalized coordinates of the system: $\vec{b}$ is the control vector; u(t) is the control input; A is the matrix of dimensionality nXn, the elements of which depend, possibly, on r parameters $\nu_1, \nu_2, \ldots, \nu_r$; ε is a small parameter.

Let us make the following assumptions:

I. The system (2.43) is a set of connected oscillators characterized by the frequency σ_i (i=1, 2,..., n), and it can be (for $u(t)\equiv 0$) both stable and unstable as a result of the effect of positional nonconservative forces.

II. The elements of the matrix B of generalized dissipative forces are small, which is characterized by the introduction of the small parameter ε.

III. The variability of the coefficients of the system (2.43) is small in the characteristic time interval $T \sim 2\pi/\sigma$.

Then let us propose that the measuring device of the control system receives the signal

$$v(t)=(\vec{g}, \vec{x}), \qquad (2.44)$$

which is a physically observable value; the vector $\vec{g}=(g_1,\ldots,g_n)$ is the observation vector for the investigated system.

The equation of the control system will be assumed in the following form:

$$L(p)u=L_1(p)v, \qquad (L)$$

where $L(p)=L_1(p)/L_0(p)$ is the transfer function of the control system given by its frequency characteristic

$$L(i\omega)=A(\omega)[\cos\varphi(\omega)+i\sin\varphi(\omega)].$$

With respect to the control system we shall assume the following:

1. The eigenvalues of the operator $L_0(p)$ belongs to the region of stability Q_z which does not intersect with the region of eigenvalues Q_D in the matrix A for all the variations of the parameters $\nu_1, \nu_2, \ldots, \nu_r$ in the given region.

2. The disturbances Δp_j (j=1,2,...,n) of the eigenvalues (p_j^2) of the matrix A caused by the effect of the control system are small in the sense that $|\Delta p_j| \ll \Omega$, where Ω is the characteristic frequency.

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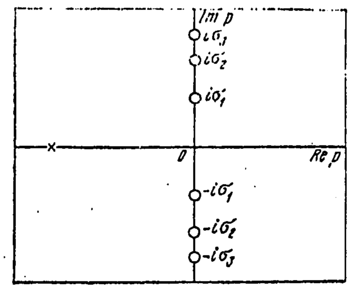

Figure 2.1. Roots of the characteristic equation of an open system:
o -- object of control; x -- control system

3. The condition sign [Im $L(i\omega_j]$=const is satisfied, where ω_j=Im p_j; p_j are the eigenvalues of the matrix A.

The condition 1 means that the control system as an element of the closed automatic control circuit is asymptotically stable, and it retains this property under all conditions of movement of the investigated system.

Fig 2.1 shows the roots of the characteristic equation of this system (2.43) including three connected oscillators characterized by the frequencies σ_1, σ_2, σ_3 (the symbols ○) ($u(t)\equiv 0$, $\varepsilon=0$). On the same figure the multiplication symbol denotes the eigenvalues of the operator $L_0(p)$ satisfying the condition 1.

The condition "2" obviously imposes restrictions on the amplification coefficient of the control systems and means that the eigenvalues of the closed system made up of the object of control and the control system are closed to their rated values (Fig 2.1) calculated in the open state of the system ($u(t)\equiv 0$).

Condition "3" is the condition of "uniformity" of the phase shifts under the effect of the operator L for all of the eigenvalues p_j of the object of control.

Fig 2.2 shows the amplitude and phase characteristics of the control system satisfying condition 3. In the given case, as is obvious:

$$\text{sign}\,[\text{Im}\,L(i\omega_j)] = +1 \quad (j=1,\ 2,\ 3).$$

Let us investigate the problem of the so-called phase [56] stabilization of the object of control (D), and accordingly let us consider the following problems.

Problem 2.1. Let the dynamic system

$$\dot{\vec{x}} = A\vec{x} + \vec{l}u, \qquad v = (\vec{g}, \vec{x}), \qquad (D)$$
$$L_0(p)u = L_1(p)v, \qquad (L)$$

be given, where the object of control (L) is defined by the properties I-III, the control system, by the properties 1-3.

It is required that the regions in which the closed system (D)-(L) will be stable for any control system satisfying the conditions 1-3 be isolated in the space of the parameters of the object (D).

Problem 2.2. Let the dynamic system be given:

$$\dot{\vec{x}} = A\vec{x} + \vec{b}u, \qquad v = (\vec{g}, \vec{x}), \qquad (D)$$
$$L_0(p)u = L_1(p)v, \qquad (L)$$

where the object of control is defined by the properties I-III, the control system, by the properties 1-2.

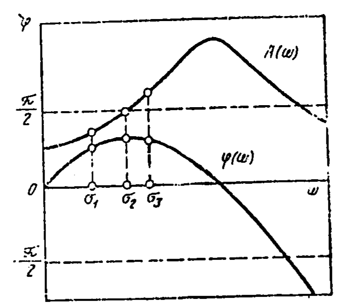

Figure 2.2. Standard phase-amplitude characteristics of the control system

What should the requirements be on the regulator as alternative (3) in order to insure stability of the system (D)-(L) in the given region of variation of the parameters of the object (D)?

Let

$$\Phi_0(p^2) = a_0 p^{2m} + a_1 p^{2(m-1)} + \ldots + a_{2(m-1)} p^2 + a_{2m} \qquad (2.45)$$

be the characteristic equation of the open system [for $u(t) \equiv 0,\ v(t) \equiv 0$]; $p_j\ (j = 1, 2, \ldots, n)$, be its roots.

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Then the case

$$p_j = -\omega_j \ (j=1, 2, \ldots, n) \qquad (2.46)$$

corresponds to the dynamically stable object (D), the case

$$p_{1,2} = \alpha \pm i\omega; \quad p_{3,4} = -\alpha \pm i\omega \qquad (2.47)$$

corresponds to its dynamic instability.

Then let us denote

$$|(p^2E - A)L_0(p) - (\vec{b}\vec{g})L_1(p)| = \Phi_0(p^2)L_0(p) + \Phi_k(p^2)L_1(p) = 0 \quad (2.48)$$

as the characteristic equation of the closed system (D)-(L);

$W(p)=\tilde{V}/\tilde{U}$ -- the transmission function of the control object;

(D); $\tilde{v}(p)$, $\tilde{u}(p)$ -- the Laplacian transform of the generalized coordinates v(t), u(t),

$\mu_k(k=1, 2,\ldots, n-1)$ -- zeroes of the transmission function W(p);

$\lambda_k(k=1, 2,\ldots, n)$ -- the ones of the transmission function W(p);

$K=(\vec{b}, A\vec{b}, \ldots, A^{n-1}\vec{b})$ -- the controllability matrix of the system (D);

$G=(\vec{g}', A'\vec{g}', \ldots, (A')^{n-1}\vec{g}')$ -- the observability matrix of the system (D).

Simultaneously with the system (D)-(L) we shall also consider the system

$$\ddot{Y} = \Lambda Y + \beta u, \ v = h'Y, \ u = L(p)v, \qquad (2.49)$$

(where $\Lambda=\mathrm{diag}\ \{\lambda_1, \lambda_2,\ldots, \lambda_n\}$) describing the movement of the dynamic system in the base made up of the eigenvectors of the matrix A.

Let us introduce the following definition.

Definition 2.1. The object of control (D) will be called stabilized, if:

a) The ones λ_k of the transmission function $W(p^2)$ of the object are prime, real and negative;

b) The zeros λ_k and the ones μ_k of the transmission function $W(p^2)$ are permutated in the following order:

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \ldots < \mu_{n-1} < \lambda_n.$$

Definition 2.2. The unstabilized object of control (D) will be called dynamically unstable if the condition "a" of the definition 2.1 is violated and structurally unstable if condition "b" is violated.

The stabilizability criterion, the formulation of which is the purpose of further analysis, is the basic instrument in solving the formulated problems (2.1) and (2.2).

2.2. Study of the Stabilizability of Oscillatory Systems with One Input

Quadratic Form $(S\vec{X}, \vec{X})$.

Stabilizability Criterion

Let us propose that the roots p^2 of the characteristic equation p_j^2 of the open system

$$\Phi_0(p^2)=a_0p^{2m}+a_2p^{2(m-1)}+\ldots+a_{2(m-1)}p^2+a_{2m}=0 \qquad (2.50)$$

are real, prime and negative (condition "a" of definition 2.1 is satisfied).

The case of dynamic instability (2.47), that is, instability of a special type, is considered separately (section 2.3, Chapter 2).

Let us introduce the stabilizability matrix $S=GK^{-1}$ associated with the controllability matrices K and observability matrices G. Let us denote also by Δ_j^S (j=1,2,..., n) the successive principal minors of the matrix $S=\|s_{ij}\|_1^n$, $V(\varkappa_1, \varkappa_2, \ldots, \varkappa_n)$ -- the number of sign changes in the series $\varkappa_1, \varkappa_2, \ldots, \varkappa_n$.

The following statements with respect to the properties of the matrix S are valid:

Property 1. The matrices S and S^{-1} are symmetric, that is,

$$S'=S;\ (S^{-1})'=S^{-1}.$$

Proof. Let us first demonstrate that on transformation of the coordinates $\vec{x}=T\vec{y}$ the matrix S varies in accordance with the equality

$$\bar{S}=T'ST. \qquad (2.51)$$

Actually, the transformation $\vec{x}=T\vec{y}$ reduces the system (D)-(L) to the form

$$\dot{\vec{y}}=\tilde{A}\vec{y}+\tilde{\vec{b}}u,\ v=(\tilde{\vec{g}}\vec{y}),\ u=L(p)v,$$

where

$$\tilde{A}=T^{-1}AT,\ \tilde{\vec{b}}=T^{-1}\vec{b},\ \tilde{\vec{g}}=T'\vec{g}.$$

Here, it is obvious that

$$\tilde{K} = T^{-1}K,\ \tilde{G} = T'G.$$

Hence, comes the correctness of (2.51):

$$\tilde{S} = \tilde{G}\tilde{K}^{-1} = T'GK^{-1}U = T'ST.$$

Let us return to the proof of property 1.

For the system (2.49) we can write:

$$\tilde{K} = \operatorname{diag}\{\beta_1, \beta_2, \ldots, \beta_n\} U_V;\ G = \operatorname{diag}\{h_1, h_2, \ldots, h_n\} U_V, \qquad (2.52)$$

where $U_V = U_V(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the Vandermondian matrix:

$$\det U_V \neq 0.$$

Consequently,

$$\tilde{S} = \tilde{G}\tilde{K}^{-1} = \operatorname{diag}\{h_1\beta_1^{-1}, h_2\beta_2^{-1}, \ldots, h_n\beta_n^{-1}\} = \tilde{S}'.$$

Then let us perform some nondegenerate transformation $\vec{Y} = T_1\vec{\zeta}$. We obtain

$$\tilde{\tilde{S}}' = (T_1'\tilde{S}T_1)' = T_1'\tilde{S}'T_1 = T_1'\tilde{S}T_1 = \tilde{\tilde{S}}.$$

Since T_1 is an arbitrary transformation, the symmetry of the matrix S is proved.

Property 2. The sign determinacy of the matrices S, S^{-1} does not depend on the base.

Proof. Let us place the quadratic form

$$\mathfrak{w} = (S\vec{x}, \vec{x})$$

in correspondence to the system (D).

Since on the basis of the equality (2.51)

$$\tilde{\mathfrak{w}} = (\tilde{S}\vec{y}, \vec{y}) = (T'ST\vec{y}, \vec{y}) = (ST\vec{y}, T\vec{y}) = (S\vec{x}, \vec{x}) = \mathfrak{w},$$

on transformation of the coordinates $\vec{X} = T\vec{y}$ the form $\mathfrak{w}$ does not change, and, consequently, its properties do not depend on the base, but are determined by the internal (structural) properties of the system (D). In particular, this pertains to the property of the sign determinacy of the form $(S\vec{x}, \vec{x})$. On the other hand, by definition, the matrix S is sign determinant if the form $\mathfrak{w}$ is sign determinant. Consequently, the property of sign determinacy of the matrix S does not depend on its specific representation in some base.

Now let us formulate the following statement -- the criterion of stabilizability.

The object of control (D) is stabilizable when and only when:

a) All the eigenvalues of the operator A are prime, real and negative;

b) The matrices S and S^{-1} are sign determinant, that is, all of their eigenvalues are simultaneously either positive or negative.

Proof. The characteristic equation of system (2.49) is $|(p^2E-\Lambda)L_0(p)-\beta h'L_1(p)|=0$,

hence $\Phi_k(p^2)=\sum_{i=1}^{n}h_i\beta_i\prod_{j\neq i}^{n}(p^2+\omega_j^2)$.

For determinacy let us set $0<\omega_1^2<\omega_2^2<\ldots<\omega_n^2$ and let us calculate the value of $\Phi_k(p^2)$ at the points $p^2=-\omega_j^2(j=1,2,\ldots,n)$. We obtain

$$\Phi_k(-\omega_m^2)=\sum_{l=1}^{n}\beta_l h_l\prod_{l\neq j}^{n}(\omega_j^2-\omega_m^2)=\beta_m h_m\prod_{j\neq m}^{n}(\omega_j^2-\omega_m^2).$$

Let us introduce the notation

$$\Phi_k(-\omega_m^2)=\beta_m h_m\Delta_m\ (m=1,2,\ldots,n).$$

Considering the alternatability of ω_j we obtain

$$\Delta_1>0;\ \Delta_2<0;\ldots;\ \Delta_n(-1)^{n+1}>0.$$

From geometric arguments it is obvious that for alternation of zeros and ones of the transmission function of the object $W(p^2)$ it is necessary and sufficient that the following inequalities be satisfied simultaneously

$$\Phi_k(-\omega_1^2)>0.\ \Phi_k(-\omega_2^2)<0,\ldots,(-1)^{n+1}\Phi_k(-\omega_n^2)>0;$$
$$\Phi_k(-\omega_1^2)<0,\ \Phi_k(-\omega_2^2)>0,\ldots,(-1)^{n+1}\Phi_k(-\omega_n^2)<0.$$

From the presented inequalities it follows that the zeros λ_k and the ones μ_k alternate when and only when

$$\operatorname{sign}(h_l\beta_l)=\text{const.}\qquad(2.53)$$

As a result of the condition (2.52) the statement is proved, for

$$\operatorname{sign}(h_l\beta_l)=\operatorname{sign}(h_l\beta_l^{-1}).$$

Note 1. In the arbitrary base the stabilizability criterion of the object (D) has the form

$$V(1,\ \Delta_1^s, \ldots,\ \Delta_n^s)=0 \qquad (2.54)$$

[as the condition of positive determinacy of the form $(S\vec{x},\ \vec{x})$] or

$$V(1,\ \Delta_1^s, \ldots,\ \Delta_n^s)=n \qquad (2.55)$$

[as the condition of the negative determinacy $(S\vec{x},\ \vec{x})$].

Note 2. From (2.52) we have

$$\det\tilde{K}=\beta_1\beta_2\ldots\beta_n\det U_V,\quad \det\tilde{G}=h_1h_2\ldots h_n\det U_V.$$

For controllability and observability of the object (D) it is necessary and sufficient that the following inequalities be satisfied

$$h_i\beta_i\neq 0 \quad (i=1,\ 2, \ldots,\ n). \qquad (2.56)$$

Comparing the conditions (2.54), (2.55) and the stabilizability condition (2.56), we note that the latter are stronger (and, consequently, more meaningful) conditions imposed on the system (D) as on the object of control.

Note 3. If

$$h_i\beta_i=0 \qquad (2.57)$$

for any $i=\ell$ the corresponding zero λ_ℓ and one μ_ℓ coincide. Thus, the condition (2.57) as the condition of uncontrollability of the object (D), in terms of the transmission function of the object of control, means cancellation of zero and one, that is, the factors $(p^2+\mu_\ell)$ and $(p^2+\lambda_\ell)$ in the numerator and the denominator of the fraction $W(p^2)$.

In conclusion let us prove the following statement.

If the investigated dynamic system is stable for any values of the parameters of the object and the control system satisfying the conditions (1)-(3), it retains this property in the region of parameters given by one of the equalities

$$V(1,\ \Delta_1^s, \ldots,\ \Delta_n^s)=0,$$
$$V(1,\ \Delta_1^s, \ldots,\ \Delta_n^s)=n;$$

in other words, when the object of control (D) is stabilizable.

Proof. When studying the small disturbances of the natural frequencies of the object of control (D) the following approximate formula exists:

$$\mathrm{Re}\, p_j = -\frac{\Phi_k(-\omega_j^2)}{\Phi_0'(-\omega_j^2)}\left[\frac{\mathrm{Im}\, L(i\omega_j)}{2\omega_j}\right] (j=1, 2,\ldots, n), \qquad (2.58)$$

which for the values of $\Phi_k(-\omega_j^2)$ $(j=1,2,\ldots, n)$ calculated by the formula (2.58) and for $\Phi_0 = \prod_{j=1}^{n}(p^2+\omega_j^2)$ gives

$$\mathrm{Re}\, p_j = \frac{1}{2}\beta_j h_j \,\mathrm{Im}\, L(i\omega_j). \qquad (2.59)$$

As a result of (2.59) and property III of the control system the last equality proves the statement, for if the system (D)-(L) is stable, then for all j obviously

$$\mathrm{Re}\, p_j < 0. \qquad (2.60)$$

The results obtained obviously solve problems (2.1) and (2.2). Indeed, the conditions (2.54)-(2.55) include only the parameters of object of control (D). On satisfaction of the conditions, that is, on realization of certain requirements on the parameters of the object of control, the closed system (D)-(L) is stable for any regulator defined by the conditions (1)-(3), which is required in the problem (2.1). On the other hand, calculating the signs of the minors $\Delta_j{}^s$ (j=1, 2,..., k), it is possible to determine for which $j=\ell$ the criterion (2.54)-(2.55) is violated and on the basis of the condition (2.59) which phase conditions $\phi(\omega_j)$ are needed in order to insure satisfaction of the conditions of stability (2.60), that is, the conditions of the problem (2.2).

The general conclusion for the performed analysis consists in the fact that the structural properties of the oscillatory system (D) with respect to the control system (L) can be investigated using the quadratic form given by the symmetric matrix $S=GK^{-1}$ on the basis of the stabilizability criterion (2.54)-(2.55) without using specific information about the parameters of the control system. This turns out to be highly significant circumstance, as is obvious from what follows, when designing oscillatory systems.

The regions of nonstabilizability obtained on the basis of the formulas (2.54)-(2.55) include (by definition) the regions of dynamic instability of the object of control (D). Here the boundaries of the regions of instability have, on the basis of condition (2.55), the property that one of two properties of the system -- controllability or observability -- is lost along them.

In order to illustrate the close relation of the mentioned properties of the dynamic system (controllability, observability, dynamic stability and stabilizability) let us again return to the example in Chapter 1 -- the controlled system of two oscillators.

Example. Let us consider the dynamic system Σ including the object of control

$$\dot{\vec{x}} = A\vec{x} + \vec{b}u;$$

$$\vec{x} = (x_1, x_2);\ A = \left\| \begin{matrix} -1 & -\beta_1 \\ -\alpha\beta_2 & -\alpha \end{matrix} \right\|;\ \vec{b}' = \left\| \begin{matrix} 1 \\ c \end{matrix} \right\|;\ \vec{g}' = \left\| \begin{matrix} 1 \\ \mu \end{matrix} \right\| \qquad (D')$$

and the control system

$$u = L(p)v,\ v = (\vec{g}, \vec{x}). \qquad (L')$$

Here β_1, β_2 are the coefficients characterizing the asymmetric position relation of two linear oscillators corresponding to the indexes 1 and 2; $\alpha>0$.

The object of control (D') turns out to be stable if

$$(1-\alpha)^2 + 4\alpha\beta_1\beta_2 < 0. \qquad (2.61)$$

It is proposed that the control system satisfies the conditions I-II, that is, the closure of the system (D') by the control system (L') does not disturb its oscillatory nature.

Let us calculate the elements of the matrices K, G, $S=GK^{-1}$:

$$K = \left\| \begin{matrix} 1 & \tau_1 \\ c & \tau_2 \end{matrix} \right\|;\ G = \left\| \begin{matrix} 1 & \varkappa_1 \\ \mu & \varkappa_2 \end{matrix} \right\|;\ S = \frac{1}{\tau_2 - c\tau_1} \left\| \begin{matrix} \tau_2 - c\tau_1 & \varkappa_1 - \tau_1 \\ \mu\tau_2 - c\varkappa_1 & \varkappa_2 - \mu\tau_2 \end{matrix} \right\| = S';$$

$$\tau_1 = -1 - \beta_1 c;\ \tau_2 = -\alpha\beta_2 - \alpha c;\ \varkappa_1 = -1 - \alpha\mu\beta_2;\ \varkappa_2 = -\beta_1 - \alpha\mu.$$

The condition of stabilizability of the object (D) has in the given case the form

$$V(1, \Delta_1^s, \Delta_2^s) = \{^0_2\}.$$

which is equivalent to the condition

$$\det S = \det(KG^{-1}) = \frac{\tau_2 - c\tau_1}{\varkappa_2 - \mu\varkappa_1} > 0. \qquad (2.62)$$

Using the system of dimensionless parameters, the inequality (2.62) is presented in the form

$$(\mu + \alpha\mu^2\beta_2 - \beta_1 - \alpha\mu)(c + \beta_1 c^2 - \alpha c - \alpha\beta_2) > 0. \qquad (2.63)$$

It is easy to see that the condition (2.63) can be obtained just as the condition of alternation of zeros and ones of the transition function of the object of control which in the given case has the form

$$W(p^2) = \gamma \frac{p^2(1+\mu c) + (\alpha + \mu c - \mu\alpha\beta_2 - \beta_1 c)}{p^4 + p^2(1+\alpha) + (\alpha - \alpha\beta_1\beta_2)}, \qquad (2.64)$$

where $\gamma > 0$ is the parameter.

Considering the following systems of equations jointly

$$\begin{cases} \beta_1\beta_2 = -\dfrac{(1-\alpha)^2}{4\alpha}, \\ \alpha\beta_2 - c^2\beta_1 - c(1-\alpha) = 0 \ (c), \end{cases} \qquad \begin{cases} \beta_1\beta_2 = -\dfrac{(1-\alpha)^2}{4\alpha}, \\ \alpha\mu^2\beta_2 + \mu(1-\alpha) - \beta_1 = 0 \ (\mu), \end{cases}$$

let us note that tangency of the hyperbola $\beta_1\beta_2 = -(1-\alpha)^2/4\alpha$ and the straight lines (c) and (μ) occurs at the points designated in Fig 2.3 by the symbols Δ and 0 respectively:

$$\left[-\frac{1-\alpha}{2c};\ \frac{c(1-\alpha)}{2\alpha}\right];\ \left[\frac{\mu(1-\alpha)}{2};\ -\frac{1-\alpha}{2\mu\alpha}\right].$$

The following cases of mutual arrangement of the boundaries of the regions of dynamic instability and structural instability of the object of control (D') are possible:

a) $c<0,\ \mu>0,\ \alpha>1$ or $c>0,\ \mu<0,\ \alpha<1$;
b) $c<0,\ \mu<0,\ \alpha>1$ or $c>0,\ \mu>0,\ \alpha<1$;
c) $c>0,\ \mu>0,\ \alpha>1$ or $c<0,\ \mu<0,\ \alpha<1$;
d) $c>0,\ \mu<0,\ \alpha>1$ or $c<0,\ \mu>0,\ \alpha<1$.

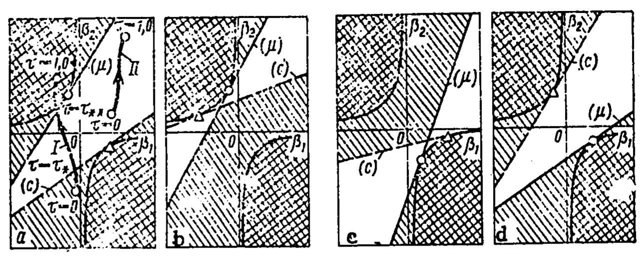

Figure 2.3. Regions of instabilizability for a controllable oscillatory system with two degrees of freedom

The inequalities (2.61), (2.63) combined with the condition of stability of the closed system (D')-(L')

$$\operatorname{Re} p_j = -\frac{\Phi_k(-\omega_j^2)}{\Phi_0'(-\omega_j^2)} \cdot \frac{\operatorname{Im} L(i\omega_j)}{2\omega_j} < 0$$

[see (2.58)] permit the following interpretation of the obtained regions to be given.

Let us consider them successively in the order corresponding to increased weight of the requirements on the characteristics of the control system:

The noncrosshatched region [system (D)] is stabilizable -- the stability of the closed system is insured by satisfaction of uniform phase conditions III:

$$\sin\varphi(\omega_1)<0,\ \sin\varphi(\omega_2)<0 \text{ or } \sin\varphi(\omega_1)>0,\ \sin\varphi(\omega_2)>0;$$

The crosshatched regions [system (D)] is nonstabilizable -- in order to insure stability of the closed system it is necessary to construct the control system so that it will realize satisfaction of nonuniform requirements

$$\sin\varphi(\omega_1)<0,\ \sin\varphi(\omega_2)>0 \text{ or } \sin\varphi(\omega_1)>0,\ \sin\varphi(\omega_2)<0$$

on the phase characteristic of the control system (L);

The regions that are double crosshatched (insurance of stability of the closed system of a dynamically unstable object by a control system of the adopted structure is impossible).

Along the rectilinear sections of the boundaries of the regions of stabilizability, the system obviously is uncontrollable or unobservable. It is easy to see that on variation of the parameters α, μ, c the boundary of the region of dynamic instability turns out to be the envelope of the family of straight lines -- boundaries of the regions of stabilizability.

The constructed regions will permit an estimate to be made of the structural properties of the object (D) in the entire range of variation of any characteristic parameter (most frequently selected as an independent parameter τ -- the time of movement).

Actually, for example, let us introduce the characteristic line $\Gamma=\Gamma(\tau)$ corresponding to the given object of control and given by the parametric equation

$$\beta_1=\beta_1(\tau),\quad \beta_2=\beta_2(\tau).$$

Let the parameter τ vary in the interval $[\tau_0, \tau_1]$. Then the position of the line $\Gamma=\Gamma(\tau)$ with respect to the regions of stabilizability [see Fig 2.3, curves I, II)] gives information about the dynamic properties of the investigated system for any values of $\tau=\tau^*$.

Let us consider the situation illustrated in Fig 2.3 (curve I). The dynamic system (D) passes successively through the segments of structural instability, structural stability, again structural instability and finally, dynamic instability on variation of τ.

For the given parameters of the control system, for example, for

$$\operatorname{sign}\{\operatorname{Im}[L(i\omega_1)]\}=\operatorname{sign}\{\operatorname{Im}[L(i\omega_2)]\}=+1, \qquad (2.65)$$

the system with $\tau_0\leqslant\tau\leqslant\tau_*$ & $\tau_{**}\leqslant\tau\leqslant\tau_1$ is unstable.

The values of the parameter $\tau=\tau_*$ and $\tau=\tau_{**}$ correspond to unobservability (uncontrollability, respectively) of the initial system, for the line $\Gamma=\Gamma(\tau)$ intersects the corresponding straight lines (μ) and (c).

If we select the parameters $\beta_1>0$, $\beta_2>0$ of the system so that the condition of dynamic stability is satisfied

$$(1-\alpha)^2+4\alpha\beta_1\beta_2>0$$

and the condition of structural stability

$$(\mu+\alpha\mu^2\beta_2-\beta_1-\alpha\mu)(c+\beta_1c^2-\alpha c-\alpha\beta_2)>0$$

in the entire interval $\tau_0\leq\tau\leq\tau_1$, then the investigated system (D)-(L) will be locally stable for all ω_j in the case of "coarse" adjustment (2.65) of the control system (L) (curve II in Fig 2.3).

In the general case the boundaries of the regions of structural instability of the object (D) in the space of its characteristic parameters $\nu_1, \nu_2, \ldots, \nu_r$ are given by the equation

$$\Delta_i^s[\nu_1(\tau),\ldots,\nu_r(\tau)]=0 \quad (i=1,2,\ldots,m);$$

the boundaries of the regions of characteristic dynamic instability, by the equations

$$\Delta_j^D[\nu_1(\tau),\ldots,\nu_l(\tau)]=0 \quad (j=1,2,\ldots,n,\ l\leqslant r),$$

where $\Delta_i{}^S$ are the principal diagonal minors of the matrix $S=GK^{-1}$, and Δ_j^D are the Hurwitz determinants for the equation (2.88). The location line

$$\Gamma(\tau)=\{\nu_1(\tau),\ldots,\nu_r(\tau)\}$$

with respect to the regions of stabilizability gives information about the structural properties of the object of control with respect to the control system (L) belonging to the given class.

From the presented arguments it follows that for solution of the problem of phase stabilization of the object of control, its parameters must be selected so as to insure first of all the dynamic stability of the object and then its structural stability.

In the specific cases it is also possible to introduce the appropriate criterion of optimization, which is illustrated below by the investigated example (see Chapter 4).

Direct Method of Investigation of the Stabilizability of the Object

In many cases when analyzing the structural properties of the object of control it is expedient to use a representation of the characteristic equation of the closed system made up of the object and the control system in the form

$$1+L(p)\frac{\Phi_k(p^2)}{\Phi_0(p^2)}=0, \qquad (2.66)$$

where

$$W(p^2)=\frac{\Phi_k(p^2)}{\Phi_0(p^2)}=\frac{b_0p^{2(m-1)}+b_2p^{2(m-2)}+\ldots+b_{2(m-1)}}{a_0p^{2m}+a_2p^{2(n-1)}+\ldots+a_{2m}} \qquad (2.67)$$

is the transmission function of the object of control.

In order to establish the fact of stabilizability (or unstabilizability) of the object it is sufficient to calculate the zeros λ_k and the ones μ_k of the function $W(p^2)$ directly and check the condition of their permutatability

$$\lambda_1<\mu_1<\ldots<\mu_{n-1}<\lambda_n. \qquad (2.68)$$

The violation of the condition (2.68) for $\lambda=\lambda_\ell$ will denote nonstabilizability of the object of control in the vicinity of the corresponding frequency $\omega_\ell=i\sqrt{\lambda_k}$.

The calculation of the roots of the equations

$$\Phi_0(p^2)=a_0p^{2m}+a_2p^{2(m-1)}+\ldots+a_{2m}=0; \qquad (2.69)$$

$$\Phi_k(p^2)=b_0p^{2(m-1)}+b_2^{2(m-2)}+\ldots+b_{2(m-1)}=0 \qquad (2.70)$$

is conveniently reduced to transformation of the determinants

$$\Phi_0(p^2)=\begin{Vmatrix} a_{11}-p^2 & a_{12} & \ldots & a_{1l} \\ a_{21} & a_{22}-p^2 & \ldots & a_{2l} \\ a_{31} & a_{32} & \ldots & a_{3l} \\ \cdot & \cdot & \cdot & \cdot \\ a_{l1} & a_{l2} & \ldots & a_{ll}-p^2 \end{Vmatrix};$$

$$\Phi_k(p^2)=\begin{Vmatrix} b_1 & a_{12} & \ldots & a_{1l} \\ b_2 & a_{22}-p^2 & \ldots & a_{2l} \\ b_3 & a_{32} & \ldots & a_{3l} \\ \cdot & \cdot & \cdot & \cdot \\ b_l & a_{l2} & \ldots & a_{ll}-p^2 \end{Vmatrix}$$

to the Frobenius form

$$\tilde{\Phi}(p^2)=\begin{vmatrix} \alpha_{11}-p^2 & \alpha_{12}\ldots\alpha_{1l} \\ 1 & -p^2 \; 0 \\ \ldots & \ldots \\ 0 \; 0 & \ldots -p^2 \end{vmatrix},$$

where the index ℓ assumes the values n and n-1 respectively.

As is known, the characteristic equation corresponding to the equation $\tilde{\Phi}(p^2)=0$ has the form

$$p^{2l}-\alpha_{11}p^{2(l-1)}-\ldots-\alpha_{1l}=0.$$

The transformation of the determinants $\Phi_0(p^2)$, $\Phi_k(p^2)$ to the Frobenius form can be realized, for example, by the Danilevskiy method, which is easily subjected tc algorithmization. As the numerical execution on the digital computer shows, the method is effective for high-order systems (n~40), and, in particular, it requires a significantly smaller number of operations than, for example, the Krylov method.

Frequency Criterion of Stabilizability of the Object of Control

Finally, let us consider the frequency interpretation of the conditions of stabilizability of an object of control.

For this purpose it is sufficient to investigate the form of the frequency characteristic of the object of control in the vicinity of the ones of the transmission function $W(p^2)$ of the object of control

$$\overset{*}{p}_1=i\Omega_1,\ \overset{*}{p}_2=i\Omega_2.$$

We have the following approximate expression for the characteristic equation of a closed system:

$$1-L(p)\times\frac{(p-p_1^0)(p-p_2^0)}{(p-\overset{*}{p}_1)(p-\overset{*}{p}_2)}=0,$$

where the following is also denoted:

$$p_1^0=i\sigma_1;\ p_2^0=i\sigma_2.$$

If we denote

$$\Delta p_1=p_1-\overset{*}{p}_1;\ \Delta p_2=p_2-\overset{*}{p}_2,$$

for the increments of Δp_i we obtain the following expressions:

$$\Delta p_1 = \frac{(p_1^* - p_1^0)(p_1^* - p_2^0)}{(p_2^* - p_1^*)} \varkappa L(i\Omega_1);$$
$$\Delta p_2 = \frac{(p_2^* - p_1^0)(p_2^* - p_2^0)}{(p_2^* - p_1^*)} \varkappa L(i\Omega_2)$$

or considering the expressions for the zeros p^0 and the ones p^*:

$$\Delta p_1 = -i\varkappa \frac{(\Omega_1 - \sigma_1)(\Omega_1 - \sigma_2)}{\Omega_2 - \Omega_1} L(i\Omega_1);$$
$$\Delta p_2 = -i\varkappa \frac{(\Omega_2 - \sigma_1)(\Omega_2 - \sigma_2)}{\Omega_2 - \Omega_1} L(i\Omega_2).$$

From the expressions for Δp_1, Δp_2 we have

$$\operatorname{Re}\Delta p_1 = -\varkappa A(\Omega_1) \frac{(\Omega_1 - \sigma_1)(\Omega_1 - \sigma_2)}{\Omega_2 - \Omega_1} \sin\varphi(\Omega_1);$$
$$\operatorname{Re}\Delta p_2 = -\varkappa A(\Omega_2) \frac{(\Omega_2 - \sigma_1)(\Omega_2 - \sigma_2)}{\Omega_2 - \Omega_1} \sin\varphi(\Omega_2).$$

In the given case the condition (2.68) of alternation of zeros and ones of the transmission function (2.67) has the form $\sigma_1<\Omega_1<\sigma_2<\Omega_2$ or $\Omega_1<\sigma_1<\Omega_2<\sigma_2$.

Let us introduce the small parameter ε into the system which characterizes the presence of dissipative forces in order to study the form of the frequency characteristic W(p) in the vicinity of the ones $i\Omega_1$, $i\Omega_2$.

Let us set

$$p_1^* = \alpha_1 + i\Omega_1; \quad p_1^0 = \mu_1 + i\sigma_1;$$
$$p_2^* = \alpha_2 + i\Omega_2; \quad p_2^0 = \mu_2 + i\sigma_2;$$
$$W(p) = \frac{(p - p_1^0)(p - p_2^0)}{(p - p_1^*)(p - p_2^*)} = 1 + W_{10}(-W_{11} + W_{12}), \qquad (2.71)$$

where

$$W_{11} = \frac{\gamma_1}{p - p_1^*}; \quad \gamma_1 = \frac{(p_1^* - p_1^0)(p_1^* - p_2^0)}{(p_2^* - p_1^*)};$$
$$W_{12} = \frac{\gamma_2}{p - p_2^*}; \quad \gamma_2 = \frac{(p_2^* - p_1^0)(p_2^* - p_2^0)}{(p_2^* - p_1^*)}.$$

Let us propose that the frequencies Ω_1, Ω_2 are different. Let us assume

$$p_2^* - p_1^* \approx i(\Omega_2 - \Omega_1).$$

For simplicity let us also set $\alpha_1=\mu_1$, $\alpha_2=\mu_2$, which does not limit the generality of the arguments.

Let us define the form of the frequency characteristics of the system, considering each of the terms in the formula (2.71) successively. It is easy to show that the function

$$W=\frac{a+ib}{p-(\alpha+i\omega)}$$

(a=const, b=const) realizes the mapping of the imaginary axis of the plane (α, ω) of the variable p into the circle of the plane (u, v) of the complex variable W passing through the origin of the coordinates and given by the equation

$$\left(u+\frac{a}{2\alpha}\right)^2+\left(v+\frac{b}{2\alpha}\right)^2=\frac{a^2+b^2}{4\alpha^2}. \qquad (2.72)$$

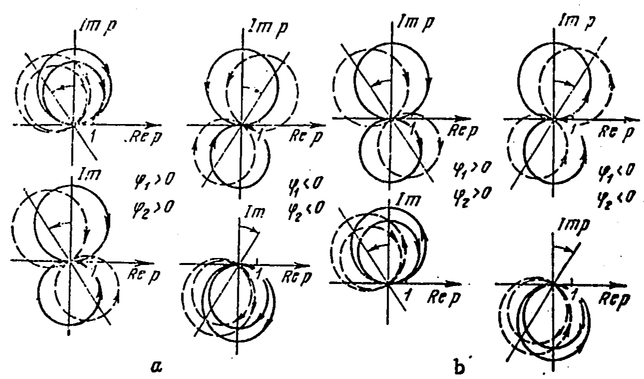

Figure 2.4. Phase-amplitude characteristics of the structurally stable (a) and structurally unstable object of control (b)

Let us find the parameters of the circles (2.72) for each of the functions

$$W_{10}W_{11}=\frac{A_1+iB_1}{p-\dot{p}_1}, \quad W_{10}W_{12}=\frac{A_2+iB_2}{p-\dot{p}_2}.$$

As a result we obtain

$$A_1=-\frac{(\alpha_1-\alpha_2)(\Omega_1-\sigma_1)}{\Omega_2-\Omega_1}; \quad A_2=\frac{(\alpha_2-\alpha_1)(\Omega_2-\sigma_2)}{\Omega_2-\Omega_1};$$
$$B_1=\frac{(\Omega_1-\sigma_1)(\Omega_1-\sigma_2)}{\Omega_2-\Omega_1}; \quad B_2=\frac{(\Omega_2-\sigma_1)(\Omega_2-\sigma_2)}{\Omega_2-\Omega_1}.$$

If we denote the slope angle of the diameter of the circle (2.72) to the v axis by θ, the following expressions are correct

$$\operatorname{tg}\theta=\frac{\Omega_1-\sigma_1}{\alpha_1-\alpha_2},\quad \operatorname{tg}\theta_2=\frac{\Omega_2-\sigma_2}{\alpha_1-\alpha_2}.$$

Since by the condition $|\alpha|<<|\Omega_2-\Omega_1|$, the angle θ is in all cases close to $\pm\pi/2$.

The phase-amplitude characteristics of the transmission function of the object of control (D) corresponding to two terms in the formula (2.71) are presented in Fig 2.4.

The phase-amplitude characteristics of the closed system (D)-(L) are the same circles as in Fig 2.4 rotated by the angles ϕ_1 and ϕ_2, respectively, the radius of which is proportional to the amplification coefficient of the regulator on the frequencies Ω_1, Ω_2.

In all cases the point (+1) is critical (in the sense of the Naiquist criterion), and the stability of the system (D)-(L) corresponds to encompassing the point (+1) by the circle gone around counterclockwise (for $\omega\to\infty$).

Therefore the structural stability of the object of control corresponds to Fig 2.4, a: the point (+1) is encompassed -- when the control system is switched on (L) -- by all the circuits gone around counterclockwise for $\omega\to\infty$, and not encompassed by the circuits running clockwise.

In Fig 2.4, b we have cases of a structurally unstable object of control: for phase lead ($\sin\phi_1>0$, i=1,2) or lag ($\sin\phi_i<0$, i=1,2) of the control system it is impossible to achieve satisfaction of the conditions of stability of the closed system: at least one circuit gone around counterclockwise for $\omega\to\infty$ does not encompass the critical point (+1) or, on the contrary, the circuit gone around clockwise encompasses the point (+1) which also leads to instability of the closed system (D)-(L).

2.3. Dynamic Instability as a Form of Nonstabilizability of the Object of Control

General Remarks

Let us make the assumption of natural dynamic instability of the object of control, and let us assume that the characteristic equation of the open system $|p^2E-A|=0$ has roots of the type $p_{1,2}=\pm\alpha+i\omega$; $p_{3,4}=\pm\alpha+i\omega$, which indicates instability characteristics of the system with positional nonconservative relations.

What occurs in this case with the system on closure of the feedback circuit by the control system of the type (L)?

The characteristic equation of the closed system

$$\dot{\vec{x}} = A\vec{x} + \vec{b}u, \qquad (D)$$

$$u = L(p)v; \quad v = (\vec{g}, \vec{x}) \qquad (L)$$

is representable in the form

$$-1 + \sum_{j=1}^{n} \frac{L(\lambda_j)\Phi_k(\lambda_j)}{\Phi_0'(\lambda_j)(p^2+\lambda_j)} + \varkappa(p), \qquad (2.73)$$

where $\varkappa(p)$ is the analytical function determined by the peculiarities of the function L(p).

Let us use the resonance properties of the investigated oscillatory system and investigate the equation (2.73) in the vicinity of the ones $p_{1,2} = \pm\alpha + i\omega$, retaining the corresponding terms in the expression (2.73).

We have

$$1 + L(p)\frac{a}{\omega^2}\,\frac{p^2 - i(\sigma_1+\sigma_2) - \sigma_1\sigma_2}{p^2 - i2\omega p - (\alpha^2+\omega^2)}, \qquad (2.74)$$

where a is a constant determining the effectiveness of the control input; $i\sigma_1$, $i\sigma_2$ are the zeros of the transmission function of the object of control which are assumed to be different.

Denoting

$$\zeta(p) = -\frac{1}{2}L(p)\frac{a}{\omega^2},$$

we obtain the equation

$$1 - 2\zeta(p)\,\frac{p^2 - i(\sigma_1+\sigma_2)p - \sigma_1\sigma_2}{p^2 - i2\omega p - (\alpha^2+\omega^2)} = 0. \qquad (2.75)$$

Then let us set

$$\zeta(i\omega) = \zeta^0(\omega)\exp[i\varphi(\omega)] = Q_1(\omega) + iQ_2(\omega)$$

and let us replace the variables $p = \mu + i(\omega + \nu)$, so that $p^2 = \mu^2 - (\Omega - \nu)^2 + 2i\mu(\omega + \nu)$.

We obtain the equation (2.74) in the following form:

$$1 - 2\zeta(i\omega)\left\{1 + \frac{i[2\omega - (\sigma_1+\sigma_2)]\lambda + [\alpha^2 - (\Omega - \sigma_1)(\Omega - \sigma_2)]}{\lambda^2 - \alpha^2}\right\} = 0. \qquad (2.76)$$

Transforming the equation (2.76) considering (2.75), we obtain the following equation with complex coefficients:

$$f(\lambda)=(1-2Q_1-2iQ_2)\lambda^2+[-i2\varkappa_1Q_1+2\varkappa_1Q_2]\lambda-$$
$$-[2\varkappa_2Q_1+i2\varkappa_2Q_2+\alpha^2-2\alpha^2Q_1-i2\alpha^2Q_2]=0, \qquad (2.77)$$

where

$$\varkappa_1=2\Omega-(\sigma_1+\sigma_2);\quad \varkappa_2=\alpha^2-(\omega-\sigma_1)(\omega-\sigma_2).$$

For stability of the closed system in the investigated frequency band obviously it is necessary and sufficient that

$$\mathrm{Re}\,\lambda_1<0;\quad \mathrm{Re}\,\lambda_2<0.$$

Let us investigate the polynomial (2.77) using the following theorem, which is a generalization of the known Routh-Hurwitz criterion [19].

If the complex polynomial f(z) is given, for which

$$f(iz)=b_0z^n+b_1z^{n-1}+\ldots+b_n+i(a_0z^n+\ldots+a_n),$$

where the polynomials $f(z)=a_0z^n+\ldots+a_n$; $f_2(z)=b_0z^n+\ldots+b_n$ are mutually prime, then the number k of the roots of the polynomial f(z) located in the right halfplane is defined by the formula

$$k=V(1,\ \nabla_2,\ldots,\ \nabla_{2n});\ \nabla_{2n}=\begin{vmatrix} a_0a_1\ldots a_{2p-1} \\ b_0b_1\ldots b_{2p-1} \\ 0a_0\ldots\ldots \\ \ldots\ldots \end{vmatrix}, \qquad (2.78)$$

where

$$p=1,\ 2,\ldots,\ n;\quad a_\kappa=b_k=0 \text{ for } k>n.$$

In accordance with the conditions of the theorem let us compare the polynomial

$$f(i\lambda)=-(1-2Q_1-2iQ_2)\lambda^2+i[-i2\varkappa_1Q_1+2\varkappa_1Q_2)\lambda-$$
$$-[2\varkappa_2Q_1+i2\varkappa_2Q_2+\alpha^2-2\alpha^2Q_1-i2\alpha^2Q_2],$$

from which we find the coefficients

$$a_0=-1+2Q_1;\quad a_1=2\varkappa_1Q_1;\quad a_2=-2\varkappa_2Q_1-\alpha^2+2\alpha^2Q_1;$$
$$b_0=-2Q_2;\quad b_1=-2\varkappa_1Q_2;\quad b_2=2\varkappa_2Q_2-2\alpha^2Q_2.$$

Setting n=2 in the formula (2.78), we obtain the condition of stability of the polynomial $f(\lambda)$ in the form $\nabla(1, \nabla_2, \nabla_4)=0$ or $\nabla_2>0$, $\nabla_4>0$.

Calculating the determinants ∇_2, ∇_4, we find

$$\nabla_2=\varkappa_1;\ \nabla_4=-4Q^2(a^2\varkappa_1^2+\varkappa_2^2).$$

As is obvious, $\nabla_2=\varkappa_1\gtrless 0;\ \nabla_4<0$, if at least one of the variables $\varkappa_1, \varkappa_2$ is nonzero.

Thus, the conditions of stability of the closed system (D)-(L) are not satisfied: the characteristic equation of the closed system in the presence of the natural dynamic instability of the object of control has roots with positive real part.

From the results of section 2.2 it follows that if the object is structurally unstable, its stabilization is theoretically impossible by "fine" adjustment of the control system on each of the natural frequencies of the system (in practice, this can far from always be realized). As is obvious, the stabilization of a dynamically unstable object by the selection of the control system parameters (L) is impossible. This fact indicates the theoretical difference between the two investigated forms of nonstabilizability of the oscillatory object -- its natural and structural instability. Therefore in addition to the criterion of stabilizability (2.54)-(2.55) it is also desirable to have the criterion of dynamic stability of the object of control as the most unfavorable form of its unstabilizability.

Sufficient Criterion of Dynamic Stability of an Object

Let us obtain a sufficient criterion of stability of a system (D) by using the theory of Cauchy indexes. In equation (2.69) let us set $p=i\omega$ and let us consider the equations

$$\Phi_0(\omega^2)=a_0\omega^{2m}-a_1\omega^{2(m-1)}+\ldots+(-1)^m a_{2m}=0. \tag{2.79}$$

It is obvious that if the equation (2.79) has prime real roots, the system (D) is dynamically stable.

Thus, the sufficient criterion of dynamic stability of the object of control appears to be possible to formulate as the criterion of realness of the roots of the equation (2.79). Let us designate by

$$I_{-\infty}^{+\infty}\frac{\Phi_0'(\omega)}{\Phi_0(\omega)}$$

the index of the Cauchy real rational function

$$R(\omega)=\frac{\Phi_0'(\omega)}{\Phi_0(\omega)}=\frac{b_0\omega^{2m-1}-b_2\omega^{2m-3}+\ldots}{a_0\omega^{2m}-a_2\omega^{2(m-1)}+\ldots}.$$

Then the necessary and sufficient condition of realness and primeness of the roots of the equation (2.79) assumes the form

$$I_{-\infty}^{\infty}\frac{\Phi_0'(\omega)}{\Phi_0(\omega)}=2m.$$

In order to calculate the index in the lefthand side, let us construct the Sturm series:

$$f_1(\omega)=\Phi_0(\omega)=a_0\omega^{2m}-a_2\omega^{2(m-1)}+\ldots+a_{2m};$$
$$f_2(\omega)=\Phi_0'(\omega)=b_0\omega^{2m-1}-b_2\omega^{2m-3}+\ldots+b_{2(m-1)};$$
$$f_3(\omega)=\frac{a_0}{b_0}\omega f_2(\omega)-f_1(\omega)=c_0\omega^{2(m-1)}-c_2\omega^{2(m-2)}+\ldots;$$
$$c_0=a_2-\frac{a_0}{b_0}b_2;\quad c_2=a_4-\frac{a_0}{b_0}b_4.$$

The coefficients of the remaining polynomials are defined analogously

$$f_4(\omega),\ldots,\ f_k(\omega).$$

Let us note that, as follows from general theory, k=2m+1, that is, the Sturm series is complete.

Let us compile the Routh table

$$\begin{matrix} a_0, & a_1, & a_2\ldots \\ b_0, & b_1, & b_2\ldots \\ c_0, & c_1, & c_2\ldots \end{matrix} \qquad (2.80)$$

On the basis of the Sturm theorem

$$I_{-\infty}^{+\infty}\frac{\Phi_0'(\omega)}{\Phi_0(\omega)}V(-\infty)-V(\infty)=2m. \qquad (2.81)$$

In the given case

$$V(\infty)=V(a_0,\ b_0,\ c_0,\ldots)\,V(-\infty)=V(a_0,\ -b_0,\ldots), \qquad (2.82)$$

where a_0, b_0, $c_0\ldots$ are the coefficients of the Routh system (2.80).

From the expression (2.82) it follows that

$$V(-\infty)+V(\infty)=2m. \qquad (2.83)$$

From equalities (2.81), (2.83) we find

$$V(a_0,\ b_0,\ c_0,\ldots)=0. \qquad (2.84)$$

By what has been proved, the criterion (2.84) is the necessary and sufficient condition of the absence of multiple roots of the equation (2.79) and at the same time the sufficient criterion of stability of the initial system.

Let us express the criterion found directly in terms of the coefficients of the polynomial $\Phi_0(p^2)$. Together with equations (2.79) let us consider the auxiliary equations

$$\varphi(p)=\eta_1\Phi_0(p)+\eta_2\Phi_0'(p)+\eta_3 p\Phi_0'(p)=0, \tag{2.85}$$

where $\eta_j>0\ (j=1,\ 2,\ 3)$.

The Hurwitz matrix for the equation (2.85) using the previously notation for the coefficients of the polynomials Φ_0 and Φ'_0 has the form

$$H=\left\|\begin{matrix} \eta_2 b_0 & \eta_2 b_2 & \ldots \\ \eta_1 a_0+\eta_3 b_0 & \eta_1 a_2+\eta_3 b_2 & \ldots \\ 0 & \eta_2 b_0 & \\ \ldots & \ldots & \ldots \end{matrix}\right\|$$

and, obviously, is equivalent to the Routh matrix

$$R=\left\|\begin{matrix} b_0 & b_2 & b_4\ldots \\ 0 & c_0 & c_2\ldots \\ 0 & 0 & d_0\ldots \\ \ldots & \ldots & \ldots \end{matrix}\right\|. \tag{2.86}$$

The equivalence of the matrices H and R will permit us to express all of the elements R (that is, the elements of the Routh system) in terms of the minors of the Hurwitz matrix H and, consequently, in terms of the coefficients of the given polynomial.

Let us designate: $\Delta_1=b_0;\ \Delta_2=bc_0;\ \Delta_3=b_0c_0d_0;\ \ldots$ so that

$$b_0=\Delta_1;\quad c_0=\frac{\Delta_2}{\Delta_1};\quad d_0=\frac{\Delta_3}{\Delta_2};\ \ldots$$

Thus, the sufficient criterion of dynamic stability (2.84) will assume the form

$$V\left(a_0,\ \Delta_1,\ \frac{\Delta_2}{\Delta_1},\ \ldots,\ \frac{\Delta_{2m}}{\Delta_{2m-1}}\right)=0 \tag{2.87}$$

or $a_0\Delta_1>0;\ \Delta_2>0;\ a_0\Delta_3>0, \ldots, a_0\Delta_{2m}>0,$ where Δ_i ($i=1,2,\ldots, 2m$) are the Hurwitz determinants for the equation (2.85).

We obtain the following result. In order that the object of control with characteristic equation (2.79) be dynamically stable, it is sufficient that the polynomial (2.35) be Hurwitz.

Setting $\eta_3=0$; $\mu=\eta_2/\eta_1$ in equation (2.85), we find

$$\varphi(p)=\Phi_0(p^2)+\mu\Phi_0'(p^2)=0. \qquad (2.88)$$

In this form the auxiliary equation (2.85) is also used later.

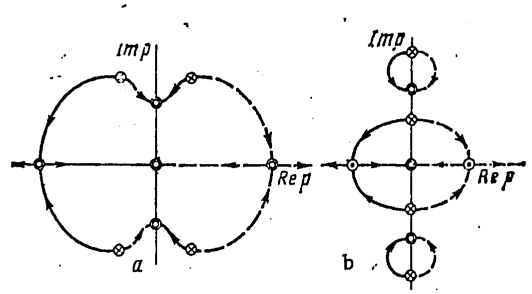

Figure 2.5. Root hodographs for the equation $\phi(\mu, p)=0$ for unstable (a) and stable object of control (b)

Example. The problem of the mutual arrangement of the roots of the polynomials $\Phi_0(p^2)$ and $\Phi_0(p^2)+\mu\Phi_0'(p^2)$ for the various values of the parameter μ is of interest. Let us illustrate the typical situation in the example of the system of section 2.2 including two bound oscillators:

$$\ddot{q}_1+\sigma_1^2 q_1+\sigma_1^2\beta_1 q_2=0;$$
$$\ddot{q}_2+\sigma_2^2 q_2+\sigma_2^2\beta_2 q_1=0.$$

Let us write the characteristic equation of the system

$$\Phi_0(p^2)=p^4+(\sigma_1^2+\sigma_2^2)p^2+\sigma_1^2\sigma_2^2(1-\beta_1\beta_2)=0$$

and the corresponding auxiliary equation (2.88)

$$\varphi(p)=p^4+4\mu p^3+(\sigma_1^2+\sigma_2^2)p^2+2\mu p(\sigma_1^2+\sigma_2^2)+\sigma_1^2\sigma_2^2(1-\beta_1\beta_2)=0. \qquad (2.89)$$

Let us construct the root hodographs for the equation (2.89), considering μ the parameter (Fig 2.5). The solid lines corresponding to variation of the parameter μ within the limits of $(0, \infty)$; the dotted lines for $(\infty<\mu<0)$.

The situation of Fig 2.5, a corresponds to the case of stability of the initial system:

$$p_{1,2}=\pm i\omega_1;\ p_{3,4}=\pm i\omega_2,$$

the situation in Fig 2.5, b corresponds to the case of instability:

$$p_i = \pm\alpha \pm i\omega \ (i = 1-4).$$

The root hodographs (see Fig 2.5) illustrate, as is obvious, not only the correspondence of the roots of the equations $\Phi_0(p^2)=0$, $\phi(p)=0$, but also their mutual arrangement, which is important for a number of practical problems.

In addition, Fig 2.5 shows that the condition of positiveness of the parameter μ is not significant. Setting μ<0 and repeating the arguments used above, we find that the stability criterion of the object of control can be represented in the form

$$V(a_0, b_0, c_0, \ldots) = 2m.$$

Thus, for analysis of the dynamic stability of the system using the sufficient criterion (2.87) it is necessary:

To calculate the coefficients of the characteristic polynomial

$$\Phi_0(p^2)=0;$$

To write the auxiliary equation

$$\Phi(p)=\Phi_0(p^2)+\mu\Phi_0'(p)=0;$$

To write the Hurwitz matrix H for the equation $\phi(p)=0$;

To check the satisfaction of one of two conditions:

$$V\left(a_0,\ \Delta_1,\ \frac{\Delta_2}{\Delta_1},\ldots,\ \frac{\Delta_{2m}}{\Delta_{2m-1}}\right)=0,\ \text{if}\quad \mu>0; \qquad (2.90)$$

$$V\left(a_0,\ \Delta_1,\ \frac{\Delta_2}{\Delta_1},\ldots,\ \frac{\Delta_{2m}}{\Delta_{2m-1}}\right)=2m,\ \text{if}\quad \mu<0. \qquad (2.91)$$

Other known stability criteria, in particular, the Lienard-Shipard and Mikhaylov criteria, in particular, can also be applied to the investigation of the equation $\phi(p)=0$. The expediency of using one criterion or another is obviously dicated by the specific situation.

2.4. Investigation of the Dynamic Instability of an Object of Control

Canonical Form of the Oscillation Equation

Without limiting the generality let us assume that the characteristic equation

$$\Phi_0(p^2)=a_0p^{2m}+a_2p^{2(m-1)}+\ldots+a_{2m}=0$$

has one group of roots of the type $p_i=\pm\alpha+i\omega$, where $\alpha>0$.

Together with equation $\Phi_0(p^2)=0$ let us also consider the equation

$$\Phi(\lambda)=a_0\lambda^m+a_2\lambda^{m-1}+\ldots+a_{2m}=0$$

with respect to $\lambda=p^2$, the roots of which are the eigenvalues of the matrix A:

$$\lambda_{\mu\Omega}=\mu\pm i\Omega;\ \lambda_3=-\varkappa_3^2;\ \ldots;\ \lambda_m=-\varkappa_m^2, \qquad (2.92)$$

where, using the former notation

$$\mu=\alpha^2-\omega^2,\ \Omega=2\alpha\omega.$$

Let us also assume that among the roots

$$-\varkappa_j^2\ (j=3,\ 4, \ldots,\ m)$$

there are no multiple roots. It is natural to isolate the following cases: 1) $-\Omega\neq 0$ (the case of different eigenvalues of the matrix A); 2) $-\Omega=0$, $\mu\neq 0$ (the case of multiple eigenvalues).

Let $\dot{\vec{y}}=B\vec{y}$ be the canonical form of the system where B is a given (real) matrix, for example:

$$B=\left\|\begin{array}{cc|c} \mu & -\Omega & 0\ldots 0 \\ \Omega & \mu & 0\ldots 0 \\ \hline 0 & 0 & -\varkappa_3^2\ldots 0 \\ \vdots & \vdots & \vdots\quad\vdots \\ 0 & 0 & 0-\varkappa_n^2 \end{array}\right\|.$$

Let T be the matrix of the desired transformation: $T=\|t_{ij}\|_1^n$.

By definition we have

$$T^{-1}AT=B \qquad (2.93)$$

or, multiplying the expressions (2.93) from the left by the matrix T and grouping terms, we obtain

$$AT-TB=0. \qquad (2.94)$$

Thus, the matrix T, if it exists, satisfies the matrix equation (2.94).

Let us introduce the vector $\vec{z}$ of dimensionality m^2:

$$z=(z_1,\ldots,\ z_{m^2})=(t_{11},\ldots,\ t_{1m},\ t_{21},\ldots,\ t_{2m},\ldots,\ t_{mn}), \qquad (2.95)$$

where the components z_k and t_{ij} are related by the expression

$$t_{ij}=z_{m(i-1)+j}\,(i,\ j=1,\ldots,\ m). \qquad (2.96)$$

Transforming equation (2.94), we obtain

$$AT=(A\times E)\vec{z};\ TB=(E\times B')\vec{z}, \qquad (2.97)$$

where AxE, ExB are the matrices $[m^2 X m^2]$ which are the Kronecker (or tensor) product of the matrices A, E and E, B', respectively, and having the following structure:

$$A\times E=\left\|\begin{array}{ccc|ccc|c|ccc}
a_{11} & \ldots & 0 & a_{12} & \ldots & 0 & \ldots & a_{1m} & \ldots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots\\
0 & \ldots & a_{11} & 0 & \ldots & a_{12} & \ldots & 0 & \ldots & a_{1m}\\ \hline
a_{21} & \ldots & 0 & a_{22} & \ldots & 0 & \ldots & a_{2m} & \ldots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots\\
0 & \ldots & a_{21} & 0 & \ldots & a_{22} & \ldots & 0 & \ldots & a_{2m}\\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \hline
a_{m1} & \ldots & 0 & a_{m2} & \ldots & 0 & \ldots & a_{mm} & \ldots & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots\\
0 & \ldots & a_{m1} & 0 & \ldots & a_{m2} & \ldots & 0 & \ldots & a_{mm}
\end{array}\right\|; \qquad (2.98)$$

$$E\times B'=\left\|\begin{array}{ccc}
B' & \ldots & 0\\
\vdots & \ddots & \vdots\\
0 & \ldots & B'
\end{array}\right\|. \qquad (2.99)$$

Considering expressions (2.95)-(2.99), it is possible to represent the equation in the following equivalent form:

$$(A\times E-E\times B')\,\vec{z}=0, \qquad (2.100)$$

where $\vec{z}$ is the vector defined by the expression (2.95).

Thus, the problem of constructing the desired transformation reduces to investigation of the linear uniform system (2.100). It is known that the eigenvalues of the matrix

$$J=A\times E-E\times B',$$

which we shall denote by $p_j^{(j)}$ $(j=1,2,\ldots,m^2)$ are related to the eigenvalues of the matrices A and B (which, by definition, coincide) by the expression

$$p^{(j)}=p_k-p_l, \qquad (2.101)$$

where the right side contains the differences of the roots with arbitrary combinations of indexes k, $\ell=1,2,\ldots,m$.

Let us return to the investigation of the isolated cases. Let

$$\lambda_{1,2}=\mu\pm i\Omega;\ \lambda_k=-\varkappa_k^2(k=3,\ 4,\ldots,\ m).$$

Considering all possible combinations of the type of (2.101) of eigenvalues of the matrix A, we find that zero is the m-tuple eigenvalue of the matrix J. The rank of the matrix J is equal to m^2-m so that there are m linearly independent vectors and the matrix T respectively.

Let $\lambda=0$ be the double eigenvalue of the matrix A. In the given case zero is the (m+2)-tuple eigenvalue of the matrix J. The rank of the matrix J is equal to m^2-m-2. Thus, the matrix T also can be constructed from the set of m linearly independent vectors.

The presented arguments give, as is obvious, the method of reducing the initial matrix A to the canonical matrix B, which from the point of view of the problems investigated later it is expedient to take in the form

$$B=\left\|\begin{array}{cc|cccc} a_{11} & a_{12} & 0 & \ldots & \ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots & \ldots & 0 \\ \hline 0 & 0 & -\varkappa_3^2 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots & -\varkappa_m^2 \end{array}\right\|, \qquad (2.102)$$

selecting the coefficients a_{11}, a_{12}, a_{21}, a_{22} so that the eigenvalues of the matrix

$$\tilde{A}=\left\|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\|$$

will be

$$\mu\pm i\Omega.$$

Actually, when selecting the matrix B in the form (2.102) the transformation $\vec{x}=T\vec{y}$ constructed by the above-discussed algorithm does not separate the physically connected partial systems (in the given case corresponding to the indexes 1 and 2), which retains the possibility of analysis of the parts of their interaction for different characteristic parameters of the object (D).

From this point of view the Jordanian form, just as certain other canonical forms (Danilevskiy, triangular) have low suitability.

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Fictitious Controllable System. Simulation of Dynamic Instability

From the results obtained in section 2.3 it follows that the analysis of the dynamic stability reduces to investigation of the equation

$$\Phi_0(p)+\mu\Phi_0'(p)=0$$

or, what amounts to the same thing, the investigation of the mutual arrangement of the zeros and ones of the transmission function

$$W(p)=\mu\frac{\Phi_0'}{\Phi_0}. \qquad (2.103)$$

Let us formulate the following problem.

Problem 2.3. Let us construct the linear dynamic system

$$\dot{\vec{z}}=F\vec{z}+\vec{\beta}u,\ u=\vec{h}'\vec{z}$$

such that:

a) The transmission function of the object $W(p)=\frac{\tilde{z}}{\tilde{u}}$ will coincide with the required

$$W(p)=\mu\frac{\Phi_0'(p)}{\Phi_0(p)};$$

b) The characteristic equation of the closed system made up of the object and the control system have the form

$$\Phi_0(p^2)+\mu\Phi_0'(p)=0. \qquad (2.104)$$

Let us denote

$$\Phi_0(p)=p^{2m}+a_2p^{2m-1}+a_4p^{2(m-2)}+\ldots+a_{2m}; \qquad (2.105)$$
$$\Phi_0(p)=b_0p^{2m-1}+b_2p^{2m-3}+\ldots+b_{2(m-1)}. \qquad (2.106)$$

The coefficients of the polynomials (2.105)-(2.106) with odd indexes are equal to zero:

$$a_1=a_{\_}=\ \ldots=a_{2m-1}=0;\ b_1=b_3=\ldots=b_{2m-1}=0, \qquad (2.107$$

where

$$b_{2j}=2(m-j)a_{2j},\ j=1,\ 2,\ldots,\ m. \qquad (2.108)$$

Let us assume that the equation (2.104) for $\mu=0$ has no multiple roots. Then on the basis of the known properties of the controlled systems the transmission function

$$W(p)=\mu\frac{b_0p^{2m-1}+b_1p^{2(m-1)}+b_2p^{2m-3}+\ldots+b_{2m-1}}{p^{2m}+a_1p^{2m-1}+a_2p^{2(m-1)}+\ldots+a_{2m}} \qquad (2.109)$$

corresponds to a real linear autonomous system, entirely controllable and entirely observable. Such a system is the only one with accuracy to linear equivalence.

Thus, the problem reduces to finding the elements of the matrices F, β, h' in some appropriately selected base.

As the first step let us find the canonical representation of the matrices F, β corresponding to the given transmission function W.

Let us consider the differential equation

$$x^{[2m]}+a_2x^{[2(m-1)]}+\ldots+a_{2m}x=b_0u^{[2m-1]}+ + b_2u^{[2m-3]}+\ldots+b_{2m-1}u \quad (2.110)$$

and let us carry out the substitution

$$\begin{aligned} &z_1(t)=x(t);\\ &z_2(t)=\dot{x}(t)-\beta_1u(t);\\ &z_3(t)=\ddot{x}(t)-\beta_1\dot{u}(t)-\beta_2u(t);\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &z_{2m}(t)=x^{[2m-1]}-\beta_1u(t)^{[2(m-1)]}-\ldots-\beta_{m-1}u(t), \end{aligned} \quad (2.111)$$

where $\beta_1, \beta_2, \ldots, \beta_{2m-1}$ are constants which must be found.

Setting

$$z_i(t)=x^{i-1}(t)-\sum_{k=0}^{i-2}u^{(k)}\beta_{i-k-1}\,(i=1,\,2,\ldots,\,2m), \quad (2.112)$$

we find the differential equation which is satisfied by the function $z_i(t)$.

From expression (2.112), considering the equations (2.111), it follows that

$$\dot{z}_i(t)=x^{(i)}(t)-\sum_{k=0}^{i-2}u^{(k+1)}(t)\beta_{i-k-1}=z_{i+1}(t)+\beta_iu(t),$$

for

$$z_{i+1}(t)=x^i(t)-\sum_{k=0}^{i-1}u^{(k)}(t)\beta_{i-k}=x^{(i)}(t)-\beta_iu(t)-\sum_{k=1}^{i-1}u^{(k)}(t)\beta_{i-k}. \quad (2.113)$$

Differentiating equation (2.113) for i-2m, we obtain

$$\dot{z}_{2m}(t)=x^{(2m)}(t)-\sum_{k=1}^{2m-1}u^{(k)}\beta_{2m-k}$$

and substituting the expression $x^{(2m)}(t)$ from equation (2.110), we find

$$x^{(2m)}(t) = -\sum_{l=0}^{2m-1} a_{2m-l}x^{(l)}(t) + b_0 u^{(2m-1)}(t) + \sum_{k=1}^{2(m-1)} b_{2m-k-l}u^{(k)}(t) + b_{2m}u(t).$$

From the expression (2.113) it follows that

$$\sum_{l=0}^{2m-1} a_{2m-l}x^{(l)}(t) = \sum_{l=0}^{2m-1} a_{2m-l}z_{l+1}(t) + \sum_{l=0}^{2m-1} a_{2m-l} \times \left[\sum_{k=1}^{l-1} u^{(k)}(t)\times\beta_{l-k}\right] + u(t)\sum_{l=0}^{2m-1} a_{2m-l}\beta_l.$$

Thus,

$$\dot{z}_{2m}(t) = -\sum_{l=0}^{2m-1} a_{2m-l}z_{l+1}(t) + b_0 u^{2m-1}(t) + \sum_{k=1}^{2(m-1)} b_{2m-k-1} + b_{2m-1}u(t) - \sum_{k=1}^{2m-1} u^{(k)}(t)\beta_{2m-k} - \sum_{l=0}^{2m-1} a_{2m-l} \times \left[\sum_{k=1}^{l-1} u^{(k)}(t)\beta_{l-k}\right] - u(t)\sum_{l=0}^{2m-1} a_{2m-l}\beta_l.$$

Considering that

$$-\sum_{l=0}^{2m-1} a_{2m-1}\sum_{k=1}^{l-1} u^{(k)}(t)\beta_{l-k} = -\sum_{k=1}^{2m-1} u^{(k)}(t)\sum_{l=1}^{2m-k-1}\beta_l a_{2m-l-k},$$

we obtain the following expression

$$\dot{z}_{2m}(t) = -\sum_{l=0}^{2m-1} a_{2m-l}z_{l+1}(t) + \sum_{k=1}^{2m-1} u^{(k)}(t) \times \left\{b_{2m-k-l} - \beta_{2m-k} - \sum_{l=1}^{2m-k-1} a_{2m-l-k}\beta_l\right\} + u(t)\left[b_{2m-1} - \sum_{l=0}^{2m-1} a_{2m-l}\beta_l\right]. \quad (2.114)$$

It is necessary to have the constant β_i so that the right side of the equation (2.114) will not depend on the arbitrary function z(t). For this purpose it is sufficient to set

$$b_{2m-k-1} = \beta_{2m-k} - \sum_{l=1}^{2m-k-1}\beta_l a_{2m-l-k} \quad (k=1, 2, \ldots, 2m-1).$$

Let us denote

$$b_{2m} - \sum_{l=1}^{2m-1} a_{2m-l}\beta_l = \beta_{2m}.$$

Then the equation for $\dot{z}_{2m}(t)$ can be represented in the form

$$\dot{z}(t) = -\sum_{l=0}^{2m-1} a_{2m-l} z_{2m-l}(t) + \beta_{2m} u(t).$$

In order to determine the constant β_i, we have the following system of equations:

$$\begin{aligned} & b_0 - \beta_1 = 0; \\ & b_1 - \beta_2 - a_1 b_1 = 0; \\ & \cdots\cdots\cdots\cdots\cdots \\ & b_{2m-1} - \beta_{2m} - a_1\beta_{2m-1} - a_2\beta_{2m-2} - \ldots - a_{2m-1}\beta_1 = 0. \end{aligned} \tag{2.115}$$

Since on the basis of (2.107) all of the coefficients of the system with odd indexes are equal to zero, $\beta_{2j}=0$ $(j=1,2,\ldots,m)$. The remaining coefficients β_{2j-1} beginning with β_1, are calculated without difficulty by successive substitutions.

In particular,

$$\beta_1 = 2m;\ \beta_3 = -2a_2;\ \beta_5 = 2a_2^2 - 4a_4;\ \ldots$$ and so on.

The system (2.115) can be written in vector form:

$$\vec{b} = D\vec{\beta},$$

where

$$D = \begin{Vmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_2 & 1 & 0 & \ldots & 0 \\ a_4 & a_2 & 1 & \ldots & 0 \\ a_6 & a_4 & a_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ a_{2m} & a_{2(m-1)} & a_{2(m-2)} & \ldots & 1 \end{Vmatrix};$$

$$\vec{b}' = \begin{Vmatrix} b_0 \\ b_2 \\ b_4 \\ b_6 \\ \vdots \\ b_{2(m-1)} \end{Vmatrix}\ \vec{\beta}' = \begin{Vmatrix} \beta_1 \\ \beta_3 \\ \beta_5 \\ \beta_7 \\ \vdots \\ \beta_{2m-1} \end{Vmatrix}.$$

It can be shown by the classical methods, for example, inversion of the matrix of coefficients $D(|D| \neq 0)$.

Thus, we have the following canonical representation of the pair of matrices F, $\vec{\beta}$ for the investigated problem:

$$\dot{\vec{z}}(t)=F\vec{z}+\vec{\beta}u; \qquad (2.116)$$
$$u=\vec{h}'\vec{z},$$

where

$$F=\left\|\begin{matrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \cdot & \dots & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & & \\ \cdot & \cdot & \cdot & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_{2m} & 0 & -a_{2(m-1)} & \dots & -a_2 & 0 \end{matrix}\right\|; \vec{\beta}'=\mu\left\|\begin{matrix} 2m \\ 0 \\ -2a_2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix}\right\|.$$

$h'=\{h_1,\dots,h_{2m}\}$ is an arbitrary vector.

Let us use the arbitrariness in the selection of the vector $\vec{h}$ to formulate the required characteristic equation (2.104) of the closed system.

On the basis of the absence of multiple roots of the equation $\Phi_0=0$ the vector $\vec{h}$ exists. Let us show that in order that the characteristic equation of the closed system have the form (2.104), it is necessary and sufficient that

$$\vec{h}'=(-1,\ 0,\ 0,\dots,\ 0).$$

Sufficiency. Let $F=\{f_1,\dots,f_n\}$; then $\beta h'=(-\beta,\ 0,\dots,0)$, and, consequently, the system (2.116) assumes the form

$$\dot{z}=(F+\beta h')z.$$

Its characteristic polynomial is

$$|F+\beta h'-Ep|=0$$

and can be written as follows:

$$|(f_1+\beta f_2,\dots,\ f_n)-Ep|=0. \qquad (2.117)$$

Expanding the determinant (2.117) with respect to the first column, we obtain the required expression for the characteristic polynomial:

$$|F-Ep|+|(\beta f_2,\dots,\ f_n)-Ep|=0,$$

or for the selected β:

$$\Phi_0(p)+\mu\Phi_0'(p)=0,$$

Q.E.D.

Necessity. Let the vector $\vec{h}$ be selected so that

$$|F+\beta h'-Ep|=\Phi_0(p)+\mu\Phi_0'(p). \quad (2.118)$$

Since the pair of matrices F, β is a controlled pair, the base exists:

$$\begin{aligned}\vec{e}_1&=F^{(n-1)}\vec{g}+a_2F^{(n-3)}\vec{g}+\ldots+a_{2(m-1)}F\vec{g};\\ \vec{e}_2&=F^{(n-2)}\vec{g}+a_2F^{(n-4)}\vec{g}+\ldots+a_{2(m-2)}F\vec{g}; \quad (2.119)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ \vec{e}_{2m-1}&=F\vec{g};\\ \vec{e}_{2m}&=\vec{g}, \quad (2.119)\end{aligned}$$

where $\vec{g}=\vec{\beta}$, with respect to which the system (2.116) has the form $\dot{\vec{y}}=F\vec{y}+\vec{\beta}u$;

$$F=\begin{Vmatrix}0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ -a_{2m} & 0 & -a_{2(m-1)} & & 0\end{Vmatrix};\ \vec{\beta}=\begin{Vmatrix}0\\0\\ \vdots\\1\end{Vmatrix}; \quad (2.120)$$

$$u=\vec{h}'\vec{y}.$$

The corresponding transformation is given by the matrix P made up of the vectors $\vec{e}_j$ (j=1,2,...,2m):

$$\vec{z}=P\vec{y}.$$

Thus,

$$\vec{h}'=\vec{h}'P;\ \vec{\beta}=P^{-1}\vec{\beta}.$$

It is obvious that the vector h' is found uniquely from the condition

$$\vec{h}'=(0;\ \beta_{2m-3},\ldots,\ 0,\ -\beta_2,\ 0,\ -\beta_0).$$

Consequently, the vector $\vec{h}'$ is also found uniquely. The preceding analysis indicates that $\vec{h}'=(-1,0,\ldots,0)$.

This statement is proved.

Thus, the solution of the above-stated problem used the following dynamic system:

$$\dot{\vec{z}}=F\vec{z}+\vec{\beta}u;\quad u=\vec{h}'\vec{z}; \quad (2.121)$$

$$F=\begin{Vmatrix}0 & 1 & 0 & 0 & \ldots & 0\\ 0 & 0 & 1 & 0 & \ldots & 0\\ \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot\\ -a_{2m} & 0 & -a_{2(m-1)} & 0 & \ldots & 0\end{Vmatrix};\ \vec{\beta}_\mu=\begin{Vmatrix}2m\\0\\ \cdot\\ \cdot\\0\end{Vmatrix};\ \vec{h}=\begin{Vmatrix}-1\\0\\ \cdot\\ \cdot\\0\end{Vmatrix}.$$

The system (2.121) describes the movement of some fictitious closed system including the given object with one input u(t) and one output $\vec{z}(t)$ with rigid feedback.

Known methods of analyzing linear systems, including the method of analyzing the structural stability investigated above, can be applied to the system (2.121). In the latter case it is sufficient to investigate the equation

$$1+\mu\frac{\Phi_0'(p)}{\Phi_0(p^2)}=0,$$

where the role of the "controlling" polynomial Φ_k is played by the polynomial Φ_0'.

In conclusion, let us find the matrix of the transformation of the initial system to the form (2.121). Let the initial system be given in the form

$$\dot{\vec{x}}=A\vec{x}+\vec{b}u,$$

where A is the matrix of dimensionality (2mX2m) characterizing the object of control; $\vec{b}$ is a vector such that the pair {A, $\vec{b}$} is controllable.

Thus, two pairs of matrices {A, $\vec{b}$}, {F, $\vec{\beta}$}, are given which correspond to the same object of control, but described in different bases. It is required that the matrix of the transition T be defined in explicit form.

The transformation T causes conversion of the matrices described by the formulas

$$\vec{\beta}=T\vec{b};\ F=TAT^{-1}. \qquad (2.122)$$

Considering $\vec{\beta}=T\vec{b}$ we have:

$$F\vec{\beta}=TAT^{-1}T\vec{b}=TA\vec{b};$$
$$F^2\vec{\beta}=TAT^{-1}TA\vec{b}=TA^2\vec{b};$$
$$\cdots\cdots\cdots\cdots$$
$$F^{(n-1)}\vec{\beta}=TAT^{-1}TA^{(n-2)}\vec{b}=TA^{(n-1)}\vec{b}.$$

Thus,

$$(\vec{\beta},\ F\vec{\beta},\ldots,\ F^{(n-1)}\vec{\beta})=T(\vec{b},\ A\vec{b},\ldots,\ A^{n-1}\vec{b}).$$

By assumption, the systems (2.121), (2.122) are controllable. Consequently, the quadratic matrices

$$Q_\beta=(\vec{\beta},\ F\vec{\beta},\ldots,\ F^{(n-1)}\vec{\beta});\ Q_b=(\vec{b},\ A\vec{b},\ldots,\ A^{(n-1)}\vec{b})$$

are not degenerate, and the inverse matrix exists $Q_b^{-1}=(\vec{b},\ A\vec{b},\ldots,\ A^{(n-1)}\vec{b})^{-1}$.

Thus, we arrive at the following results. The transformation matrix $\{A, \vec{b}\} \Rightarrow \{F\vec{\beta}\}$ is defined by the formula

$$T=(\vec{\beta},\ F\vec{\beta},\ldots,\ F^{(n-1)}\vec{\beta})(\vec{b},\ A\vec{b},\ldots,\ A^{(n-1)}\vec{b})^{-1}.$$

Example. Let the oscillatory system including two oscillators be described by the equation

$$\begin{Vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Vmatrix} = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -\beta_1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta_2\alpha & 0 & -\alpha & 0 \end{Vmatrix} \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix}; \qquad (2.123)$$

where

$$x_1 = q_1;\ x_2 = \frac{\dot{q}_1}{\sigma_1};\ x_3 = q;\ x_4 = \frac{\dot{q}_2}{\sigma_2};$$

$$\alpha = \sigma_2^2/\sigma_1^2.$$

In the given case

$$\Phi_0(p^2) = p^4 + a_2p^2 + a_4,$$

where

$$a_2 = 1 + \alpha;\ a_4 = \alpha(1 - \beta_1\beta_2).$$

The canonical representation of the system (2.123) has the form

$$\begin{Vmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{Vmatrix} = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & 0 & -a_2 & 0 \end{Vmatrix} \begin{Vmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Vmatrix} + \mu \begin{Vmatrix} 4 \\ 0 \\ -2a_2 \\ 0 \end{Vmatrix} u, \qquad (2.124)$$

$$u = \vec{h}'\vec{z};\ \vec{h} = (-1,\ 0,\ 0,\ 0).$$

Let us find the transition matrix T. We have

$$Q_b = \begin{Vmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\beta_2\alpha \\ 0 & 0 & -\beta_2\alpha & 0 \end{Vmatrix}; \quad Q_b^{-1} = \begin{Vmatrix} 0 & 1 & 0 & -\frac{1}{\beta_2\alpha} \\ 1 & 0 & -\frac{1}{\beta_2\alpha} & 0 \\ 0 & 0 & 0 & -\frac{1}{\beta_2\alpha} \\ 0 & 0 & -\frac{1}{\beta_2\alpha} & 0 \end{Vmatrix};$$

$$Q_\beta = \begin{Vmatrix} 2 & 0 & -a_2 & 0 \\ 0 & -a_2 & 0 & a_2^2 - 2a_4 \\ -a_2 & 0 & a_2^2 - 2a_4 & 0 \\ 0 & a_2^2 - 2a_4 & 0 & 3a_2a_4 - a_2^3 \end{Vmatrix}.$$

Thus,

$$T=\left\|\begin{matrix} 0 & 2 & 0 & \left[-\frac{2}{\beta_2\alpha}+\frac{a_2}{\beta_2\alpha}\right] \\ -a_2 & 0 & \left[\frac{a_2}{\beta_2\alpha}+\frac{2a_4-a_2^2}{\beta_2\alpha}\right] & 0 \\ 0 & -a_2 & 0 & \left[\frac{a_2}{\beta_2\alpha}+\frac{-a_2^2+2a_4}{\beta_2\alpha}\right] \\ a_2^2-2a_4 & 0 & \left[-\frac{a_2^2-2a_4}{\beta_2\alpha}-\frac{3a_2a_4-a_2^3}{\beta_2\alpha}\right] & 0 \end{matrix}\right\|.$$

The desired transformation has the form

$$\vec{z}=T\vec{x},$$

where T is the matrix.

The canonical system corresponds to the transmission function

$$W(p)=2\mu\frac{2p^3+p(1+\alpha)}{p^4+p^2(1+\alpha)+(\alpha-\alpha\beta_1\beta_2)} \tag{2.125}$$

and the characteristic equation of the closed system

$$\Phi(p)=p^4+4\mu p^3+(1+\alpha)p^2+2\mu(1+\alpha)+(\alpha-\alpha\beta_1\beta_2)=0.$$

Thus, the dynamic instability criterion of the system (2.123) is expressed as the condition of alternation of zeros and ones of the transmission function $W(p^2)$. The fictitious controlled system in the form of (2.121) can be used to simulate the dynamic instability of the system (D), that is, to estimate the effect on the stability of the various factors, primarily the parameters of the object of control itself.

In this chapter a study is made of the linear oscillatory systems which are objects of control with one input. Among all the systems of this class the stabilized systems are isolated characterized by the ordered arrangement of the zeros and ones of the transmission function of the object of control W(p):

$$\lambda_1<\mu_1<\ldots<\mu_{n-1}<\lambda_n.$$

The condition of permutatability of the zeros λ_k and the ones μ_k is also taken as the definition of stabilizability of the object of control (D).

Formally, the stabilizability criterion is obtained as follows. For the oscillatory system

$$\ddot{\vec{x}}=A\vec{x}+\vec{b}u,\ v=\vec{g}'\vec{x} \tag{α}$$

the controllability and observability matrices K and G are calculated, respectively:

$$K = \|\vec{b}, A\vec{b}, \ldots, A^{n-1}\vec{b}\|; \; G = \|\vec{g}', A'\vec{g}', \ldots, (A^{n-1})'\vec{g}'\|.$$

The matrix

$$S = GK^{-1} = \|s_{ij}\|_1^n,$$

as is demonstrated, is symmetric and, consequently, is a quadratic form $(S\vec{x}, \vec{x})$.

The criterion of positive (negative) determinacy of the form $(S\vec{x}, \vec{x})$ is the chain of inequalities

$$V(1, \Delta_1^s, \ldots, \Delta_n^s) = 0 \text{ for } (S\vec{x}, \vec{x}) > 0 \qquad (\beta)$$

or

$$V(1, \Delta_1^s, \ldots, \Delta_n^s) = n \text{ for } (S\vec{x}, \vec{x}) < 0, \qquad (\gamma)$$

which also serves as the criterion of stabilizability of the object of control (α), in other words, the criterion of "phase" stabilization of the object of control (D), equipped with the control system (L) of given structure.

CHAPTER 3. STABILIZABILITY OF SPACE VEHICLES

3.1. Mathematical Models of Space Vehicles (Movement in the Active Segment)

General Characteristic of the Object of Control

Let us consider in general features the structural peculiarities of flight vehicles with liquid-propulsion rocket engines as objects of control by [35, 56]. Figures 3.1 and 3.2 show two different flight vehicles: the booster rocket for space vehicles (Fig 3.1 -- "Saturn-5") and the space vehicle itself (Fig 3.2) designed for operations in open space. The fuel reserves are required for maneuvers en route to the destination planet or in the satellite orbit. The control of the flight vehicle with respect to the center of masses is realized by creating controlling moments as is obvious from Fig. 31, by deflection of the peripheral sustainer engines (the first and second stages -- Fig 3.1, a) or deflection in the two stabilization planes of the control engine and antisymmetric deflection of the special control engines for creating a heeling moment (third stage, Fig 3.1, b). The analogous control systems are also characteristic of space vehicles.

As is obvious, an important characteristic feature of the composite flight vehicle system is the presence of supporting (or suspension) fuel tanks executed in the form of thin-walled shells. The lower compartment, connected through the supporting feet and the intertank compartments which take significant compressive loads are usually more powerful structures reinforced by developed supporting framing.

The forces from the sustainer engines usually are transmitted to the hull through a quite rigid girder structure supported on a ring which plays the role of a reinforced frame member. The fuel mains running to each liquid-propellant rocket engine begin near the pole of the corresponding bottom, and the end with entry into the pump for the oxidizing engine or the combustible fuel component. At the ends of the fuel lines there are bellows which unload the walls of the lines from additional loads connected with the deformations of the hull and their natural curvature deformations. The enumerated facts have important significance when selecting an efficient dynamic hull system.

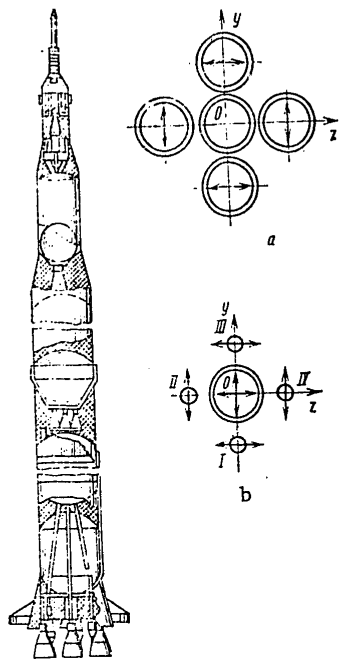

Figure 3.1. Standard composite booster rocket system of space vehicles and artificial earth satellites

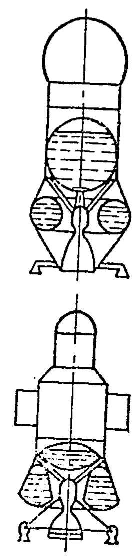

Figure 3.2. Standard composite space vehicle systems with two fuel tanks

The nature of movement of the investigated objects in the active segment of flight has no less important significance for the selection of the system. Figures 3.3 and 3.4 show examples of two characteristic types of active segments: acceleration from artificial earth satellite orbit and landing on a planet without an atmosphere. The movement in the active segment can be represented in the form of programmed movement with sufficiently smoothly varying parameters on which additional movement is proposed characterized by "small" deviations of the corresponding parameters from their programmed values. As a result, independently of the method of assigning the program the true movement differs little from the program movement.

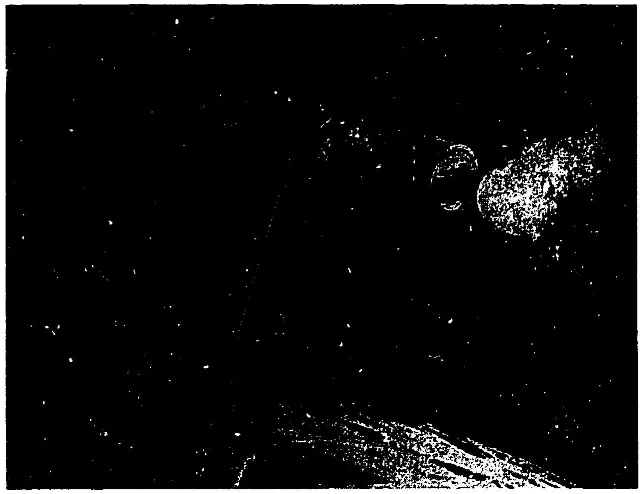

Figure 3.3. Active segment of the flight of a space vehicle: acceleration from artificial earth satellite orbit

Figure 3.4. Active segment of the flight of a space vehicle: landing on the destination planet

However, this situation is complicated by the fact that if in the program movement the hull of the space vehicle can be interpreted as a rigid solid state of variable mass and sometimes simply as a particle (of variable mass), then in the disturbed movement (usually having an oscillatory nature) it is necessary to take into account the mobility of the fuel in the tanks and the lines and also the elasticity of the structural elements of the hull. This generates problems connected with the mathematical description of the hull as an object of control, which, as is obvious, is a complex deformed system.

The role of the control system is usually played either by the automatic stabilization system (movement in the yawing, pitch, heel channels), or a liquid-fuel rocket engine (movement in the direction of the longitudinal axis of the hull of the space vehicle).

Let us note that the completeness of the mathematical model of the space vehicle must correspond to the level of development of the object (the drawings or, for example, the final determination of the parameters of the control system. It is clear that when selecting the model it is necessary to take into account all the factors which play a significant role in the given specific problem and at the same time do not overload the calculated model with various details making it unjustifiably complicated.

The level of complexity of the problem can be demonstrated in the following example. Let us assume that it is necessary to investigate disturbed motion of the booster stage of a space vehicle having four fuel components in the yawing or pitch plane. Let us take into account one degree of freedom in the liquid in each of the four tanks, two degrees of freedom of the rocket as a solid state, the first two forms of elastic vibrations of the hull and also one degree of freedom of the deflected sustainer engine.

All of this leads to an 18th-order system of differential equations (nine degrees of freedom) not taking into account the order of the control operator, the effective investigation of which is far from elementary (even in numerical form).

At the present time the development of an adequate mathematical model of the planned objects with liquid-propellant rocket engines is one of the central problems in the overall problem of insuring their dynamic stability.

This problem still does not permit complete formalization and requires the heuristic approach. The guiding idea here is usually the investigation of the frequency spectrum of the space vehicle as the object of control and comparison of it with the pass band of the automatic stabilization system and the engine, closing the system.

In Fig 3.5 as an example we have the frequency spectra of all three stages of the "Saturn-5" booster rocket falling in the range from 0 to 15 hertz. This range corresponds to the pass band of the automatic stabilization system and the liquid-fuel rocket engine and must be represented by the corresponding oscillatory elements in the dynamic system. If the filtering properties of the automatic stabilization system and the liquid-fuel rocket engine as control systems are such that on a frequency of $f>f_{max}$ (in the given case f_{max}=15 hertz) the system in practice opens, then this insures adequateness of the simplified dynamic system of the object when analyzing the stability of the closed system made up of the object and the control system. The choice of one system or another is determined by the goal of the investigation.

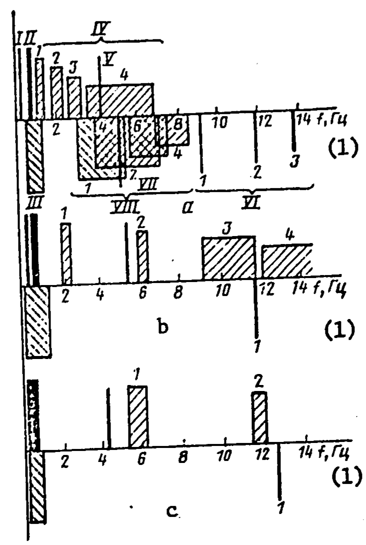

Figure 3.5. Standard low-frequency spectrum of the frequencies of the booster rocket ("Saturn-5"):
a -- first stage; b -- second stage; c -- third stage;
I -- rocket as a rigid solid state with automatic stabilization system in the yawing (pitch) plane; II -- rocket as a rigid solid state with automatic stabilization system in the heel plane;
III -- fuel components in the tanks of the separating section (first tone); IV -- hull (1, 2, 3 -- transverse oscillations; 3, 4 -- tone number); V -- sustainer engine; VI -- hull, torsional vibrations (1,2,3 -- tone numbers); VII -- liquid in the oxidizing agent fuel line; VIII -- hull, longitudinal vibrations (1,2,3,4 -- tone numbers).

Key:
1. hertz

For analysis of the static controllability reserves it is possible (in the preliminary step) to limit ourselves to a model of an absolutely rigid solid state.

When selecting the characteristics of the devices which limit the mobility of the fuel in the tanks, this is insufficient, and it is necessary to consider the wave movements on the free surface of the fuel as a source of disturbing moments for the space vehicle as a whole. The choice of efficient placement of the sensors of the space vehicle stabilization system usually is made considering the elasticity of its hull.

Thus, on the modern level of development of rocket engineering the sphere of applicability of the hypothesis of "hardening" of the liquid or "absolute rigidity" of the hull has a limited nature. When investigating the dynamic stability of the rocket fully, usually the mentioned factors are taken into account.

One of the important peculiarities of the investigated class of objects is comparative smallness of the dissipative forces during oscillations of the fuel in the compartments and the elastic vibrations of the hull. The fact that the work of these forces is small for the characteristic period permits calculation of the corresponding frequencies and forms of the vibrations without considering damping.

Finally, as for variability of physical characteristics of the space vehicle, by the statement, it is related to the consumption of the fuel in the active segment and in comparison with the characteristic oscillation period ($T \sim 0.1$ to 1.0 seconds) it is obviously small.

Summing up what has been discussed, it is possible to conclude that with respect to the general characteristics, a space vehicle as an object of control satisfies the hypotheses used when discussing the theory of stabilizability in the preceding chapter.

Control Systems: Automatic Stabilization System and Liquid-Propellant Rocket Engines

The input information for the automatic stabilization system is mismatch of the program and the realized values of the generalized coordinates of the space vehicle, and for the liquid-propellant rocket engine, variation of the pressures at the input to the pump for the oxidizing agent and the combustible fuel component, and the output information is the deviations of the controlling elements or variation of the thrust, respectively.

The automatic stabilization system and the liquid-propellant rocket engine are designed so that under test bench conditions they are dynamically stable. The spectra of the corresponding frequencies do not intersect with the spectrum of the natural frequencies of the object of control (to avoid undesirable resonances).

The actual frequency-amplitude and frequency-phase characteristics of the automatic stabilization system and the liquid-propellant rocket engine, being quite smooth, have the properties that the oscillation frequencies of the fuel and other vibrations of the hull on closure by the control system vary little (the frequency-amplitude characteristic of the control system is limited by the corresponding constant as the upper bound).

Fig 3.6 shows the functional diagrams of the closed systems made up of the object of control -- the automatic stabilization system (movement in the yawing plane) and the object of control -- the liquid-propellant rocket engine (movement in the direction of the longitudinal axis).

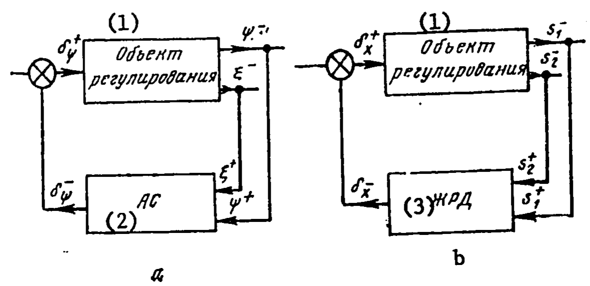

Figure 3.6. Functional diagrams of the closed system made up of the object and the control system:
a -- when moving in the yawing plane; b -- when moving in the direction of the longitudinal axis

Key:
1. object of control
2. automatic stabilization system
3. liquid-propellant rocket engine

The angular stabilization systems in the pitch plane (UST), the yawing plane (USR), and the heeling plane (USK) insure closeness of the true angular position of the hull to the program position, the normal stabilization system (NS) and the lateral stabilization system (BS), closeness of the true movement of the center of masses (metacenter) to the programmed movement. Finally, the liquid-propellant rocket engine does not interfere with the stability of the program movement in the direction of the longitudinal axis in the case of stability of the closed system.

The superscript "+" in Fig 3.6 corresponds to the input signal, the superscript "-" corresponds to the output signal.

The automatic stabilization system and the engine as control systems can be described by the following equations:

$$\delta_\vartheta + \delta_\vartheta^* = L_1'(\vartheta^0) + L_2'(\eta^0); \qquad (3.1)$$
$$\delta_\psi = L_1'(\psi^0) - L_2'(\zeta^0);$$
$$\delta_\varphi = L_3'(\varphi^0); \qquad (3.2)$$
$$\delta_x = L_1(S_1^0) + L_2(S_2^0),$$

where L_n(n=1,2), L_n' (n=1,2,3) are nonlinear operators permitting linearization in the required frequency range; $\delta_x=\Delta P/P^*$ is the increment in the thrust of the sustainer engine reduced to the normal thrust at the given altitude; $s_n^0=(P/P^*)_n$ is the pressure increment at the input through the pump for the n-th component of the fuel reduced to the static pressure of the same component; $\vartheta^0, \psi^0, \varphi^0, \eta^0, \zeta^0$ are the angular and linear mismatches between the true and programmed position of the hull at the point x0 of placement of the sensors of the UST, USR, USK, NS, and BS systems. The asterisk corresponds to the program value of the parameter.

The simplest linearized equations (3.1) have the form:

$$c_2(\ddot{\delta}_\vartheta + \ddot{\delta}_\vartheta^*) + c_1(\dot{\delta}_\vartheta^* + \dot{\delta}_\vartheta) + \delta_\vartheta^* + \delta_\vartheta = a_0\vartheta^0 + a_1\dot{\vartheta}^0 + b_0\eta^0 + b_1\dot{\eta}^0; \qquad (3.3)$$
$$c_2\ddot{\delta}_\psi + c\dot{\delta}_\psi = a_0\psi^0 + a_1\dot{\psi}^0 - b_0\zeta^0 - b_1\dot{\zeta}^0; \qquad (3.4)$$
$$e_2\ddot{\delta}_\varphi + e_1\dot{\delta}_\varphi + \delta_\varphi = d_0\varphi^0 + d_1\dot{\varphi}^0. \qquad (3.5)$$

In the frequency range corresponding to the first tone of the vibrations of the liquid in the tanks and the first two to three tones of the transverse elastic vibrations of the hull, the NS and BS systems in practice have no influence on the stability of the system (the spectra of the mentioned frequencies are separated).

Then, setting $L_2'=0$, we find that all the equations of (3.1) have identical structure and, for example, the equation corresponding to the yawing plane can be written in the form

$$\delta = L(\psi^0), \qquad (3.6)$$

omitting the nonessential indexes. The operator L in the simplest case is a piecewise linear operator so that intead of the equation (3.4), we obtain

$$c_2\ddot{\delta} + c_1\dot{\delta} + \delta = a_0\psi^0 + a_1\dot{\psi}^0. \qquad (3.7)$$

Fig 3.7 shows the standard frequency-amplitude and frequench-phase characteristics $(\tilde{\delta}/\tilde{\psi}^0)$ corresponding to the automatic stabilization system described by the equation (3.7) for the following values of the parameters:

$$a_0=2{,}1;\ a_1=1{,}26;\ c_2=0{,}00976;\ c_1=0{,}126.$$

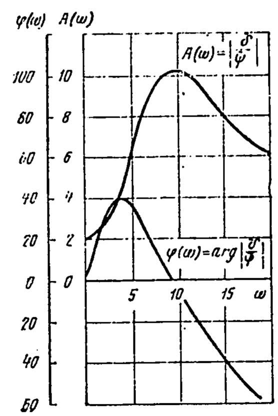

Figure 3.7. Standard frequency-amplitude and frequency-phase characteristics of the automatic angular stabilization system

The simplest linearized equations describing the dynamics of the liquid-propellant rocket engine of the open system have the form

$$\begin{aligned}\dot{q}_1+\beta_1 q_1&=a_{11}s_1^0+a_{13}\delta_x;\\ \dot{q}_2+\beta_2 q_2&=a_{22}s_2^0+a_{23}\delta_x;\\ \dot{\delta}_x+\beta_3\delta_x&=a_{31}q_1+a_{32}q_2,\end{aligned} \quad (3.8)$$

where $q_n=(\mu/\mu^*)_n$ are the increments of the mass consumption of the fuel components per second reduced to the corresponding rated values of the engine (n=1,2).

Usually the frequencies of the natural vibrations of the liquid in the combustible fuel component and oxidizing agent lines differ sharply as a result of the difference in length of these lines. Therefore the relation between the fluctuations of the pressures p_1 and p_2 is relatively weak. This provides the basis in equations (3.8) for obtaining either $s_1^0=0$ or $s_2^0=0$ depending on the situation.

After excluding q_n the equation (1.8) acquires the form

$$\delta=L_n(s_n^0)\ \ (n=1,2), \quad (3.9)$$

where L_n is in the simplest case also a piecewise linear operator

$$c_{2n}\ddot{\delta}+c_{1n}\dot{\delta}+\delta=a_{0n}s_n^0\,(n=1,2). \quad (3.10)$$

Here

$$c_{2n}=\frac{1}{\beta_n\beta_3-a_{3n}a_{n3}};\ c_{1n}=\frac{\beta_3+\beta_n}{\beta_n\beta_3-a_{3n}a_{n3}};\ a_{0n}=\frac{a_{nn}}{\beta_n\beta_3-a_{3n}a_{n3}}. \quad (3.11)$$

If we exclude not q_2 or q_1 but δ_x from the equations of the engine for $s_1^0 \equiv 0$ or $s_2^0 \equiv 0$, we arrive at the dynamic boundary condition at the lower end of the corresponding line of the type

$$q_n = W_n(p) s_n^0, \qquad (3.12)$$

where p is the differentiation operator with respect to t or considering (1.8)

$$c_{2n}\ddot{q}_n + c_{1n}\dot{q}_n + q_n = a_{0n}(s_n^0 + \beta_3 s_n^0). \qquad (3.13)$$

This boundary condition permits determination of the presented damping coefficient of the vibrations of the liquid in the line.

Fig 3.8 shows the dimensionless frequency-amplitude and frequency-phase characteristics $(\tilde{\delta}/\tilde{s}_2^0)$ and $(\tilde{q}_2/\tilde{s}_2^0)$ for hypothetical open and closed system (with respect to one of the fuel components) liquid-propellant rocket engines:

$$\frac{\tilde{\delta}}{\tilde{s}_2} = A(\omega)\, e^{i\varphi(\omega)}; \quad \frac{\tilde{q}_2}{\tilde{s}_2^0} = D(\omega)\, e^{i\psi(\omega)}. \qquad (3.14)$$

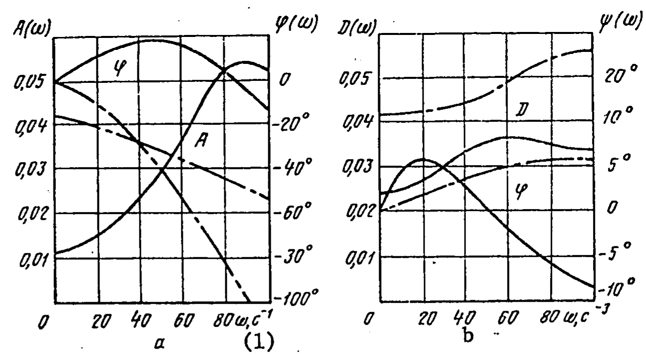

Figure 3.8. Standard frequency-amplitude and frequency-phase characteristics of closed (____) and open (-.-) system liquid-propellant rocket:
a -- relative thrust as a function of the relative pressure at the pump input; b -- relative consumption as a function of the relative pressure at the pump input.

Key:
1. $\sec^{-1}$

The second fuel component is of great interest, for it corresponds to the long line with lower frequency of the natural oscillations of the liquid.

Generalizing what has been discussed, it is possible to conclude that the control system (automatic stabilization system or liquid-propellant rocket engine) satisfies the assumptions made in Chapter 2 with respect to its characteristics.

Equations of Disturbed Motion of a Space Vehicle in the Active Segment

Fig 3.9, a shows the diagram of the transition from the initial structural design to the adopted calculation model for longitudinal oscillations of the rocket hull with tandem component (series arrangement of the fuel tanks) [56].

Fig 3.9, b depicts the equivalent elastic rod with compartments containing liquid, the two upper ones of which correspond to the rigid suspension tank on elastic couplings permitting displacement of the tanks along the axis of the hull, and the lower one, the supporting cylindrical tanks connected with a rod through the reinforced framing.

The final dynamic diagram is presented in Fig 3.9, c in which the equivalent masses on elastic couplings are indicated which simulate the masses of the liquid in the tanks and engine and also the elastic and rigid elements of the rods corresponding to the liquid in the oxidizing agent and combustible fuel component lines.

Fig 3.10 shows a diagram of the transition to the calculated model of the same initial structural design as in Fig 3.9 except with transverse oscillations.

Here the rigid compartments with the liquid rotating as a result of the presence of couplings together with certain cross sections of the hull are introduced. Then the transition is made to a uniform system of the beam type with distributed and lumped parameters. The wave components of the fuel in the tanks are simulated by certain equivalent pendulums, and the engine, by a pendulum with an additional elastic coupling. The calculation diagram is selected analogously for torsional vibrations.

In the final analysis the use of the above-mentioned mechanical models leads to the following equations of disturbed (with respect to the programmed motion) movement of the space vehicle:

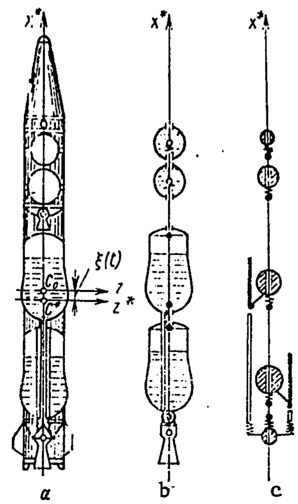

Figure 3.9. Dynamic diagram of a booster rocket during longitudinal oscillations:
a -- initial structural design with deformed (as shells) supporting tanks and rigid suspension; b -- simplified mechanical analog; c -- equivalent elastic rod

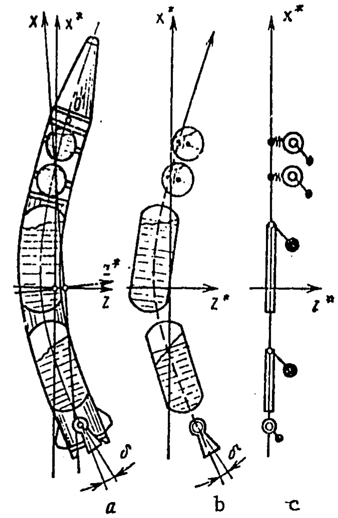

Figure 3.10. Dynamic diagram of the booster rocket during transverse oscillations:
a -- initial structural design with deformed (as a beam) supporting tanks and rigid suspension tanks; b -- simplified mechanical analog; c -- equivalent elastic rod

Transverse oscillations:

$$
\begin{gathered}
(m^0+m)\ddot{\zeta}+\sum_{n=1}^{k}\lambda_n\ddot{\zeta}_n=P_z;\\
(J^0+J)\ddot{\psi}+\sum_{n=1}^{k}(\lambda_{0n}\ddot{s}_n+\beta_{0n}s_n)=M_y;\\
a_j(\ddot{q}_j+B_{qj}\dot{q}_j+\sigma_j^2 q_j)+\sum_{n=1}^{k}(\lambda_{jn}\ddot{s}_n+\beta_{jn}\dot{s}_n)=Q_j; \qquad (3.15)\\
\mu_n(\ddot{s}_n+\beta_{s_n}\dot{s}_n+\omega_n^2 s_n)+\lambda_n\ddot{\zeta}+\lambda_{0n}\ddot{\psi}+\beta_{0n}\dot{\psi}+\sum_{j=1}^{m}(\lambda_{jn}\ddot{q}_j+\beta_{jn}\dot{q}_j)=0\\
(j=1, 2,\ldots, m;\ n=1, 2,\ldots, k).
\end{gathered}
$$

Longitudinal oscillations:

$$(m^0+m)\ddot{\xi}+\sum_{n=1}^{n}\lambda_n\ddot{s}_n=P_k;$$
$$a_j(\ddot{q}_j+\beta_{q_j}\dot{q}_j+\sigma_j^2 q_j)+\sum_{n=1}^{k}\lambda_{jn}\ddot{s}_n=Q_j; \qquad (3.16)$$
$$\mu_n(\ddot{s}_n+\beta_{s_n}+\omega_n^2 s_n)+\lambda_n\ddot{\xi}+\sum_{j=1}^{m}\lambda_{jn}\ddot{q}_j=0$$
$$(j=1, 2,\ldots, m;\ m=1, 2,\ldots, k).$$

Torsional vibrations:

$$(J^0+J)\ddot{\varphi}=M_x; \qquad (3.17)$$
$$a_j(\ddot{q}_j+\beta_{q_j}\dot{q}_j+\sigma_j^2 q_j)=Q_j\ (j=1,2,\ldots m).$$

In equations (3.15)-(3.17) the generalized coordinates ζ, ψ, ϕ, q_j, s_n have the following meaning: ζ is the deflection of the metacenter x_G of the space vehicle in the disturbed movement; ψ is the angle of deflection of the space vehicle as a solid state; ϕ is the angle of deflection of the space vehicle during oscillations with respect to the longitudinal axis; q_j are the transverse and longitudinal displacements of the cross section x=0 [equations (3.15)-(3.16)] and the angles of rotation of this cross section [equations (1.17)]; s_n are the "z-coordinates" of the free surface of the fuel in the n-th compartment.

The physical meaning of the coefficients of equations (3.15)-(3.17) is clear from their structure: (m^0+m), (J^0+J) are the apparent mass and the connected moment of inertia of the hull of the space vehicle; μ_n, a_j are the apparent masses characterizing the oscillations of the fuel in the compartment and the elastic oscillations of the hull with frequencies of σ_n and ω_j, respectively; λ_n, λ_{0n}, λ_{jn} are the coefficients of the cross couplings between the generalized coordinates s_n, q_j, ζ, ψ, β_{qj}, β_{sn}, β_{jn} are the damping coefficients.

The equations (3.15)-(3.16) describing the space vehicle of an object of control must be simplified by the equations describing the operation of the automatic stabilization system [the angular (US) and lateral (BS)], and it is also necessary to consider their active disturbances.

1. The long-period movement defined by the disturbances $P_z(t)$, $M_y(t)$, $M_x(t)$ and the properties of the space vehicle, just as a solid, absolutely rigid body:

$$\ddot{\zeta}+a_{\zeta\delta}u=P_z(t);$$
$$\ddot{\psi}+a_{\psi\delta}u=M_y(t); \qquad (3.18)$$
$$u=L_1'(\psi)-L_2'(\zeta);$$
$$\ddot{\varphi}+a_{\varphi\delta}u_\varphi=M_x(t); \qquad (3.19)$$
$$u_\varphi=L_3'(\varphi).$$

2. A short-period movement in the frequency range of the natural oscillations of the liquid in the tanks:

$$\ddot{\zeta}+a_{\zeta\psi}\psi+\sum_{n=1}^{k}a_{\zeta s_n}\ddot{s}_n+a_{\xi\delta}u=0;$$
$$\ddot{\psi}+\sum_{n=1}^{k}(a_{\psi s_n}\ddot{s}_n+a'_{\psi s_n}\dot{s}_n)+a_{\psi\delta}u=0; \qquad (3.20)$$
$$\ddot{s}_n+\beta_{s_n}\dot{s}_n+\omega_{s_n}^2 s_n+a_{s_n\zeta}\ddot{\zeta}+a_{s_n\psi}\ddot{\psi}+a'_{s_n\psi}\dot{\psi}=0;$$
$$n=1,2,\ldots,k;$$
$$u=1(\psi);$$
$$\ddot{\varphi}+\beta_\varphi\dot{\varphi}+\sum_{n=k_1+k}(a_{\varphi r_n}\ddot{r}_n+a'_{\varphi r_n}\dot{r}_n)+a_{\varphi\delta}u=0;$$
$$\ddot{r}_n+\beta_{r_n}\dot{r}_n+\omega_{r_n}^2 r_n+a_{r_n\varphi}\ddot{\varphi}+a'_{r_n}\dot{\varphi}=0; \qquad (3.21)$$
$$u_\varphi=L'(\varphi).$$

3. The short-period movement in the frequency range of the natural elastic oscillations of the hull:

$$\ddot{\zeta}+a_{\zeta\psi}\psi+a_{\zeta\delta}u=0;$$
$$\ddot{\psi}+a'_{\psi\delta}\ddot{\delta}+a_{\psi\delta}u=0; \qquad (3.22)$$
$$\ddot{q}_j+\beta_{q_j}\dot{q}_j+\omega_{q_j}^2 q_j+a_{q_j\delta}u=0;$$
$$u=L(\psi^0);$$
$$\psi^0=\psi-\sum_{j=1}^{m}\eta_j'(x^0)q_j \ (j=1,2,\ldots,m);$$
$$\ddot{\varphi}+\beta_\varphi\dot{\varphi}+a_{\psi\delta}u=0;$$

$$\ddot{p}_j+\beta_{p_j}\dot{p}_j+\omega_{p_j}^2 p_j+a_{p_j\delta}u_\varphi=0; \qquad (3.23)$$
$$\ddot{\delta}_\varphi+\beta_\delta\dot{\delta}_\varphi+\omega_\delta^2 u_\varphi-\omega_\delta^2 u_\varphi^0=0;$$
$$u_\varphi^0=L'(\varphi^0);$$
$$\varphi^0=\varphi+\sum_{j=1}^{m_1}\eta_j(x^0)p_j \ (j=1,2,\ldots,m).$$

To supplement the systems (3.15)-(3.23) let us present a more complete system taking into account the joint effect of the mobility of the fuel in the compartments and the elasticity of the hull and also the elasticity of the engine suspension (the transverse oscillation regime):

$$\ddot{\zeta}+a_{\zeta\psi}\psi+\sum_{n=1}^{k} a_{\zeta s_n}\ddot{s}_n+a_{\zeta\delta}u=\overline{P}_z(t);$$

$$\ddot{\psi}+\sum_{n=1}^{k}\left(a_{\psi s_n}\ddot{s}_n+a'_{\psi s_n}\dot{s}_n\right)+a'_{\psi\delta}\ddot{u}+a_{\psi\delta}u=\overline{M}_y(t);$$

$$\ddot{q}_j+\beta_{q_j}\dot{q}_j+\omega^2_{q_j}q_j+\sum_{n=1}^{k}\left(a_{q_j s_n}\ddot{s}_n+a_{q_j}\dot{s}_n\right)+a'_{q_j u}\ddot{u}+a_{q_j u}u=Q_j(t); \qquad (3.24)$$

$$\ddot{s}_n+\beta_{s_n}\dot{s}_n+\sigma^2_n s_n+a_{s_n\zeta}\ddot{\zeta}+a_{s_n\psi}\ddot{\psi}+a'_{s_n\psi}\dot{\psi}+\sum_{j=1}^{m}\left(a_{s_n q_j}\ddot{q}_j+a_{s_n q_j}\dot{q}_j\right)=0;$$

$$\ddot{u}+\beta_u\dot{u}+\omega^2_u u+a'_{u\psi}\ddot{\psi}+\sum_{j=1}^{m}a'_{uq_j}\ddot{q}_j-\omega^2_u u^0=0;$$

$$u^0=L'_1(\psi^0)-L'_2(\zeta^0);$$

$$\psi^0=\psi-\sum_{j=1}^{m}\eta'_j(x^0)q_j;\ \zeta^0=\zeta-(x^0-x_G)\psi+\sum_{j=1}^{m}\eta_j(x^0)q_j$$

$$(j=1,2,\ldots,m;\ n=1,2,\ldots,k).$$

Let us note that the entire set of coefficients of the system (3.24) is expediently divided into three groups:

The coefficients characterizing the space vehicle as a solid, rigid body of variable mass:

$$a_{\zeta\psi},\ a_{\zeta\delta},\ a_{\psi\delta},\ a'_{\psi\delta},\ \overline{P}_z(t),\ \overline{M}_y(t);$$

The coefficients characterizing the mobility of the liquid fuel:

$$a_{\zeta s_n},\ a_{s_n\zeta},\ a_{\psi s_n},\ a_{s_n\psi},\ \sigma^2_n,\ \beta_{s_n},\ a'_{\psi s_n},\ a'_{s_n\psi},\ x_G;$$

The coefficients characterizing the elasticity of the hull:

$$a_{q_j u},\ a'_{q_j u},\ a'_{uq_j},\ \omega^2_{q_j},\ a_j,\ \beta_{q_j},\ \beta_u,\ \overline{Q}_j(t).$$

The expressions for the enumerated coefficients depend on the compositional system of the space vehicle, and they are deciphered below for a number of important cases. The system of equations of disturbed movement of the space vehicle in the direction of the longitudinal axis also has the form analogous to (3.18-3.24):

$$\ddot{\xi}+\beta_\xi\dot{\xi}+\sum_{n=1}^{2}a_{\xi s_n}\ddot{s}_n-a_{\xi u}u=\overline{P}_\xi(t);$$
$$\ddot{q}_j+\beta_{q_j}\dot{q}_j+\omega_{q_j}^2q_j+\sum_{n=1}^{2}a_{q_js_n}\ddot{s}_n-a_{q_ju}u=\overline{Q}(t);$$
$$\ddot{s}_n+\beta_{s_n}\dot{s}_n+\omega_{s_n}^2s_n+a_{s_n\xi}\ddot{\xi}+\sum_{j=1}^{m}a_{s_nq_j}\ddot{q}_j=0; \qquad (3.25)$$
$$r_n=a_{r_ns_n}\beta_{s_n}\dot{s}_n+a_{r_ns_n}\omega_{s_n}^2s_n=0;$$
$$u=\sum_{n=1}^{2}L_n(r_n)\ (n=1,2;\ j=1,2,\dots,m),$$

where u is the increment of thrust of the engines reduced to the rated thrust at the given point of the trajectory; m is the number of forms of elastic longitudinal oscillations of the hull taken into account; L_n are the operators describing the dynamics of the engine. The generalized coordinates q_j (j=1,2,...,m) and s_n (n=1,2) have, however, another meaning than system (3.24) (see below and also section 3.5).

Let us note in conclusion that the equations of motion of the space vehicle are also used below (see Chapter 5) in matrix form:

$$M\ddot{\vec{x}}+\varepsilon B\dot{\vec{x}}+C\vec{x}=\vec{b}u, \qquad (3.26)$$
$$u=L_1(\vartheta)-L_2(\omega). \qquad (3.27)$$

Here the (k+m+2)-dimensional vectors

$$\vec{x}=\{z,\ \psi,\ s_1,\dots,\ s_k,\ q_1,\dots,\ q_m\};$$
$$\vec{b}=\{-a_{zu},\ -a_{\psi u},\ 0,\dots,\ 0,\ -a_{q_1u},\dots,\ -a_{q_mu}\}$$

have the meaning of the vectors of state and control for the system (1.25); the vectors

$$\vec{g}_\psi=\{0,\ 1,\ 0,\dots,\ 0,\ -\eta_j',\dots,\ -\eta_m'\};$$
$$\vec{g}_z=\{1,\ -x^0+x_G,\ 0,\dots,\ 0,\ -\eta_1,\dots,\ -\eta_m\}$$

are the observation vectors corresponding to the angular (ψ) and translational (z) movement of the space vehicle.

The time functions

$$v=(\vec{g}_\psi,\ \vec{x});\ w=(\vec{g}_z\vec{z})$$

are the output parameters for the object of control (3.26).

The matrices A, B and C of dimensionality nXn, where n=k+m+2, are defined directly by the coefficients of the system (1.24) or the systems analogous to it. Let us note here that the matrix A is always in this case positively defined on the basis of the positive determinacy of the quadratic form corresponding to the kinetic energy of the space vehicle.

Then, by necessity setting $L_1(v)\equiv 0$ or $L_2(w)\equiv 0$, let us consider the structural properties of the system (3.26) in the modes of translational and angular movements of the space vehicle and also in the longitudinal oscillation mode.

3.2. Simplest Cases of Investigation of the Structural Stability of the Space Vehicle

Elastic Oscillations of the Hull of the Space Vehicle

Let us investigate the stabilizability of the space vehicle in the short-period movement mode in the frequency band of the elastic oscillations of the hull. Let us consider the system (3.22), setting $L_2(w)\equiv 0$:

$$\begin{aligned} \ddot{z} &= a_{z\psi}\psi + a_{zu}u; \\ \ddot{\psi} &= a_{\psi u}u; \\ \ddot{q}_j &= -\omega_j^2 q_j + a_{q_j u}u; \end{aligned} \qquad (3.28)$$

$$\begin{aligned} \psi^0 &= \psi - \sum_{j=1}^{m} \eta_j' q_j; \\ u &= L(p)\psi^0 \ (j=1, 2, \ldots, m). \end{aligned} \qquad (3.29)$$

As is obvious from equation (3.29), it is proposed that the measuring device generates a signal $\psi^0 = \psi - \sum_{j=1}^{m} \eta_j' q_j$, in the process of movement, which is the superposition of the forms of the oscillations of the components (3.28) of the partial systems.

Let us proceed in system (3.28) to the variable

$$\psi^0 = \psi - \sum_{j=1}^{m} \eta_j' q_j. \qquad (3.30)$$

Differentiating expression (3.30) and substituting the following expression in equations (3.28)

$$\ddot{q}_j = -\omega_j^2 q_j + a_{q_j u}u,$$

we obtain the following system:

$$\ddot{z} = a_{z\psi}\psi + a_{z\psi} \sum_{j=1}^{m} \eta_j' q_j + a_{zu}u;$$

$$\begin{aligned}&\ddot{\psi}=\sum_{j=1}^{m}\eta_j'\omega_j^2 q_j+\left[a_{\psi u}-\sum_{j=1}^{m}\eta_j' a_{q_j u}\right]u;\\&\ddot{q}_j=-\omega_j^2 q_j \qquad +a_{q_j u}u;\\&\qquad\qquad j=1,2,\ldots,m\\&u=L(p)\psi\end{aligned} \tag{3.31}$$

(the index "0" is omitted for simplification).

The characteristic equation of the system (3.31) is representable in the form

$$1+L(p)\frac{\Phi_k(p^2)}{\Phi_0(p^2)}=0, \tag{3.32}$$

where $\Phi_0(p^2)=p^4\prod_{j=1}^{m}(p^2+\omega_j^2);$

$$\Phi_k(p^2)=\begin{vmatrix} p^2 & 0 & 0 & 0 & \ldots & 0\\ 0 & a_{\psi u} & \eta_1' p^2 & \eta_2' p^2 & \ldots & \eta_n' p^2\\ 0 & a_{q_1 u} & p^2+\omega_1^2 & 0 & \ldots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & a_{q_m u} & 0 & 0 & \ldots & p^2+\omega_m^2\end{vmatrix}.$$

Let us denote $c_j=\frac{a_{q_j u}}{a_{\psi u}},\ j=1,2,\ldots,m,\ a_{\psi u}\neq 0.$

We omit the factor p^2 which is common for the polynomials $\Phi_k(p^2)$ and $\Phi_0(p^2)$ (only the angular movement is taken into account).

In accordance with the definition, the system (3.31) is structurally stable if the zeros λ_k and the ones μ_k satisfying the equations $\Phi_0(\lambda_k)=0$, $\Phi_k(\mu_k)=0$, respectively, of the transmission function

$$W(p^2)=\frac{\Phi_k(p^2)}{\Phi_0(p^2)},$$

are permutated, and it is structurally unstable in all other cases.

In the given case there is no necessity for writing out the criterion of structural stability in general form, for the corresponding conditions can be obtained directly. Actually, setting $p=i\omega$ in the expression for Φ_0 and Φ_k, we obtain the expressions

$$\Phi_0(\omega^2)=\omega^2\prod_{j=1}^{m}(-\omega^2+\omega_j^2);$$

$$\Phi_k(\omega^2)=a_{\psi u}\left\{\prod_{j=1}^{m}(-\omega^2+\omega_j^2)+\sum_{j=1}^{m}\prod_{l\neq j}c_j\eta_j'\omega^2(-\omega^2+\omega_l^2)\right\}.$$

Also assuming that the frequencies ω_j are numbered in increasing order, let us calculate the values of the polynomial $\Phi_k(\omega^2)$ at the points ω_j^2 (j=1,2,...,m):

$$\Phi_k(\omega_j^2)=a_{q_j u}\eta_j'\omega_j^2\prod_{l\neq j}^{m}(-\omega_j^2+\omega_l^2).$$

For structural stability of the investigated system it is necessary and sufficient that the signs of $\Phi_k(\omega^2)$ alternate on going from number j to number j+1:

$$\Phi_k(\omega_1^2)>0;\ \Phi_k(\omega_2^2)<0;\ \Phi_k(\omega_3^2)>0;\ldots$$

or

$$\Phi_k(\omega_1^2)<0;\ \Phi_k(\omega_2^2)>0;\ \Phi_k(\omega_3^2)<0.$$

Let us denote $\Phi_k(\omega_j^2)=a_{q_j u}\Delta_j\ (j=1,2,\ldots,m),$

where $\Delta j=\omega_j^2\prod_{j=1}^{m}(-\omega_j^2+\omega_l^2).$

Since $\Delta_1>0;\ \Delta_2<0;\ \Delta_3>0;\ \Delta_4<0;\ldots$ the system is structurally stable if

$$V_{q_j\eta_j'}=a_{q_j u}\eta_j'>0\ (j=1,2,\ldots,m) \qquad (3.33)$$

or

$$V_{q_j u}\eta_j'=a_{q_j u}\eta_j'<0\ (j=1,2,\ldots,m). \qquad (3.34)$$

The boundary of the region of structural stability of the spacecraft is defined by the equation

$$V_{q_j\eta_j'}=0. \qquad (3.35)$$

The condition (3.35) denotes uncontrollability and unobservability of the system (3.28) simultaneously.

Physically, for the investigated system the condition (3.35) means either absence of information about the form $\eta_\ell'(x^0)$ of elastic vibrations of the hull ($\eta_\ell'=0$) or the absence of the possibility of control input to the partial system characterized by the number ℓ ($a_{q_\ell u}=0$).

The practical conclusions which follow from (3.33)-(3.34) can be represented in different form depending on whether L(p) is a simulating or actual operator. If $L(i\omega)=A(\omega)\exp[i\phi(\omega)]$, in the first case it is necessary to interpret the conditions (3.33) as requirements on the phase characteristic of the control system (L) following from the conditions of stability of the closed system made up of the object and the control system.

In the second case varying the sign of $(V_{q_j\eta_j}'V_{\varphi j})$ to the opposite for any number j indicates instability on the corresponding partial frequency ω_ℓ.

Rigid Space Vehicle with N Cylindrical Fuel Compartments

Let us consider the equations (3.15) where we used the controlling force $b_{zu}u$, the moment $b_{\psi u}u$, and also the projection of the following force P (the thrust of the liquid-propellant rocket engine) on the axis ζ as the generalized forces. Neglecting the elasticity of the space vehicle hull, we have:

$$(m^0+m)\ddot{z}+\sum_{i=1}^{N}\lambda_i\ddot{s}_i+P\psi=b_{zu}u;$$
$$(J^0+J)\ddot{\psi}+\sum_{i=1}^{N}\lambda_{0i}\ddot{s}_i=b_{\psi u}u; \tag{3.36}$$
$$\mu_i(\ddot{s}_i+\sigma_i^2 s_i)+\lambda_i\ddot{z}+\lambda_{0i}\ddot{\psi}=0,\ i=1,2,\ldots,N;$$
$$u=L(p)\psi. \tag{3.37}$$

In the given case the vectors $\vec{b}$, $\vec{g}$ have the form

$$\vec{b}=(b_{zu},\ b_{\psi u},\ 0,\ldots,0),\ \vec{g}=(0,1,\ldots,0).$$

Let us consider the case of coinciding partial frequencies $\sigma_1^2=\ldots=\sigma_N^2=\sigma^2$.

The characteristic equation of the system (3.36)-(3.37) has the form

$$\Phi_0(p^2)+L(p)\Phi_k(p^2)=0, \tag{3.38}$$

где $\Phi_0(p^2)=p^4(p^2+\sigma^2)^N-\nu_1 p^6(p^2+\sigma^2)^{N-1}\nu_2 p^8(p^2+\sigma^2)^{N-2}+$
$+\nu_3\sigma^2p^4(p^2+\sigma^2)^{N-1};$

$$\Phi_k(p^2)=b_{\psi u}[-p^2(p^2+\sigma^2)^N+\nu_4 p^4(p^2+\sigma^2)^{N-1}];$$

$$\nu_1=\sum_{i=1}^{N}\left[\frac{\lambda_i^2}{\mu_i(m^0+m)}+\frac{\lambda_{0i}^2}{\mu_i(J^0+J)}\right];\quad \nu_2=\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{(\lambda_{0j}\lambda_i-\lambda_{0i}\lambda_j)^2}{\mu_i\mu_j(m^0+m)(J^0+J)}; \tag{3.39}$$

$$\nu_3=\frac{P}{\sigma^2}\sum_{i=1}^{N}\frac{\lambda_{0i}\lambda_i}{\mu_i(m^0+m)(J^0+J)};\quad \nu_4=\sum_{i=1}^{N}\left[\frac{\lambda_i^2}{\mu_i(m^0+m)}+\frac{b_{zu}\lambda_i\lambda_{0i}}{\mu_i b_{\psi u}(m^0+m)}\right].$$

Excluding the factor $d(p^2)=p^2(p^2+\sigma^2)^{N-2}$ common for Φ_0 and Φ_k from the investigation, we write the characteristic equation (3.38) in the form

$$p^2(a_0p^4+a_1p^2+a_2)+L(p)(b_0p^4+b_1p^2+b_2)=0; \quad (3.40)$$
$$a_0=1-\nu_1+\nu_2;\ a_1=\sigma^2(2-\nu_1+\nu_3);\ a_2=\sigma^4(1+\nu_3);$$
$$b_0=b_{\psi u}(-1+\nu_4);\ b_1=\sigma^2 b_{\psi u}(-2+\nu_4);\ b_2=-\sigma^4 b_{\psi u}.$$

For determinacy let us assume that $-1+\nu_4>0$, $b_{\psi u}<0$. The criterion of structural stability of the system (3.36) has the form

$$\Delta_4=(a_1b_2-b_1a_2)(a_0b_1-a_1b_0)-(b_0b_2-b_0a_2)^2>0$$

or considering (3.39)

$$\nu_1\nu_4-\nu_4^2-\nu_2+\nu_3\nu_4-\nu_3\nu_4^2>0. \quad (3.41)$$

Let us introduce the dimensionless parameters

$$z_i=\frac{1}{l}\frac{\lambda_{0i}}{\lambda_i};\quad k=\frac{\lambda_i^2\mu_1}{\lambda_1^2\mu_i};\quad z_0=-\frac{b_{zu}}{lb_{\psi u}};\quad \zeta=\frac{R_0}{l\bar\sigma^2};$$
$$\gamma=\frac{\lambda_1^2}{\mu_i(m^0+m)};\quad l^2=\frac{J^0+J}{m^0+m}. \quad (3.42)$$

Here R_0 is the characteristic dimension of the compartment, $\bar\sigma$ is the dimensionless frequency corresponding to the dimensional partial frequency of the oscillations of the liquid in the cavities σ; ℓ is the radius of inertia of the body-liquid system.

Considering (3.42) the criterion of structural stability (3.41) of the system (3.36) is reduced to the form

$$\Psi=c^2(k_1Z_1+\ldots+k_NZ_N)\left\{k_1\left(Z_1+\frac{c^2-1}{2c}\right)^2+\ldots+k_N\left(\frac{c^2-1}{2c}+\frac{\zeta}{2}\right)^2-\right.$$
$$\left.-(k_1+\ldots+k_N)\left(\frac{c^2+1}{2c}+\frac{\zeta}{2}\right)^2+c\gamma\zeta\left[k_1Z_1+\ldots+k_NZ_N-\frac{k_1+\ldots+k_N}{c}\right]^2\right\}>0. \quad (3.43)$$

If we introduce the N-dimensional space, taking the values of $Z_1, Z_2,\ldots,Z_N$ as the coordinate axes, the boundaries of the regions of structural stability defined by equation $\Psi=0$ are a hyperplane $k_1Z_1+\ldots+k_1Z_N=0$ passing through the origin of the coordinates and the hyperellipsoid

$$k_1\left(Z_1+\frac{c^2+1}{2c}+\frac{\zeta}{2}\right)^2+\ldots+k_N\left(Z_N+\frac{c^2-1}{2c}+\frac{\zeta}{2}\right)^2-(k_1+\ldots+k_N)\times$$
$$\times\left(\frac{c^2+1}{2c}+\frac{\zeta}{2}\right)^2+c\gamma\zeta\left[k_1Z_1+\ldots+k_NZ_N-\frac{k_1+\ldots+k_N}{c}\right]^2\Big\}=0,$$

the center of which is shifted with respect to the origin of the coordinates to the point $(Z_1^0, Z_2^0,\ldots, Z_n^0)$, and the axes are rotated with respect to the principal axes $Z_1, Z_2,\ldots,Z_N$.

In conclusion, let us note that a characteristic feature of the investigated problem is the fact that the operator L(p) does not disturb N-2 partial systems (the system (3.36) is uncontrollable with respect to the N-2 oscillators); therefore the term "structural stability" essentially refers to the equation (3.40), and not to the initial equation (3.36).

Oscillations of the Space Vehicle in the Heel Channel

Let us consider equations (3.21) describing the small oscillations of the space vehicle with respect to the longitudinal axis under the assumption that the fuel tanks are divided by radial baffles into N compartments. Neglecting, just as before, the terms proportional to the generalized velocities having order ε, we obtain

$$\ddot{\theta}+\sum_{l=1}^{N} a_{\theta s_l}\ddot{s}_l=a_{\theta u}u;\qquad (3.44)$$
$$\ddot{s}_l+\omega_l^2 s_l+a_{s_l\theta}\ddot{\theta}=0,\ l=1,2,\dots,N;$$
$$u=L(p)\,\theta.\qquad (3.45)$$

In the system (3.44) the vector b=($a_{\theta u}$, 0..., 0), where $a_{\theta s_l}$, $a_{s_l\theta}$ are the coefficients which depend on the connected moment of inertia of the liquid, the density of the liquid, the configurations of the compartments and other parameters.

Let us note that for L(p)=0 the system (3.44) is conservative. Hence,

$$a_{s_l\theta}a_{\theta s_l}>0\ (l=1,2,\dots,N).$$

The characteristic equation of the system (3.44) will be represented in the form (3.32). Setting p=iω, we find:

$$\Phi_0(\omega^2)=\begin{vmatrix} -\omega^2 & -a_{\theta s_1}\omega^2 & -a_{\theta s_2}\omega^2 & \dots & -a_{\theta s_N}\omega^2 \\ -a_{s_1\theta}\omega^2 & -\omega^2+\omega_1^2 & 0 & \dots & 0 \\ -a_{s_2\theta}\omega^2 & 0 & -\omega^2+\omega_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_{s_N\theta}\omega^2 & 0 & 0 & \dots & -\omega^2+\omega_N^2 \end{vmatrix};$$

$$\Phi_k(\omega^2)=\begin{vmatrix} -a_{\theta u} & -a_{\theta s_1}\omega^2 & -a_{\theta s_2}\omega^2 & \dots & -a_{\theta s_N}\omega^2 \\ 0 & -\omega^2+\omega_1^2 & 0 & \dots & 0 \\ 0 & 0 & -\omega^2+\omega_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & -\omega^2+\omega_N^2 \end{vmatrix}.$$

On the basis of the definition, the system (3.44) is structurally stable if the zeros and ones of the transmission function $W(p^2)$ alternate.

Obviously, this condition will be satisfied if the signs of the function $\Phi_0(\omega^2)$ at the points $\omega^2=\omega_i^2$ (i=1,2,...,N) alternate.

We have

$$\Phi_0(\omega_i^2)=a_{\theta s_i}a_{s_i\theta}\omega_i^4(\omega_1^2-\omega_i^2)\ldots(\omega_{i-1}^2-\omega_i^2)(\omega_{i+1}^2-\omega_i^2)\ldots(\omega_N^2-\omega_i^2). \quad (3.46)$$

On the basis of conservativeness of the system (3.44) $a_{\theta s_i}a_{s_i\theta}>0$.

Assuming that $\omega_1^2<\omega_2^2<\ldots<\omega_N^2$, we find

$$\operatorname{sign}\Phi_0(\omega_i^2)=-\operatorname{sign}\Phi_0(\omega_{i+1}^2),\ i=1, 2,\ldots, N.$$

Thus, the system (3.44) is structurally stable. This means that if the model (3.44) is taken for the space vehicle, its oscillations with respect to the longitudinal axis will be stabilized if we realize the phase lead $\phi(\omega_i)>0$ of the control system in the frequency range of $\omega_1, \omega_2, \ldots, \omega_N$.

3.3. Stabilizability for the Classical System (Two Fuel Compartments)

Dynamic Stability Criterion

Let us consider the space vehicle of the classical system in detail, taking into account the presence of two types with combustible fuel component and oxidizing agent required for operation of the rocket engine. In the system of equations (3.36) we set i=2:

$$\begin{aligned}
&(m^0+m)\ddot{z}+\lambda_1\ddot{s}_1+\lambda_2\ddot{s}_2+P\psi=b_{zu}u;\\
&(J^0+J)\ddot{\psi}+\lambda_{01}\ddot{s}_1+\lambda_{02}\ddot{s}_2=b_{\psi u}u;\\
&\mu_1(\ddot{s}_1+\sigma_1^2s_1)+\lambda_1\ddot{z}+\lambda_{01}\ddot{\psi}=0; \qquad (3.47)\\
&\mu_2(\ddot{s}_2+\sigma_2^2s_2)+\lambda_2\ddot{z}+\lambda_{02}\ddot{\psi}=0;\\
&u=L(p)\Psi. \qquad (3.48)
\end{aligned}$$

The characteristic equation of the system (3.47), setting $\sigma_1^2=\sigma_2^2=\sigma^2$, will be reduced to the form

$$\Phi_0(p^2)=p^4\{[1-(\nu_1+\nu_2)+\nu_3]p^4+\sigma^2[2-(\nu_1+\nu_2)+(\nu_{1\zeta}+\nu_{2\zeta})]p^2+ +\sigma^4[1+(\nu_{1\zeta}+\nu_{2\zeta})]\}=0. \quad (3.49)$$

Here

$$\nu_1=\frac{\lambda_1^2}{\mu_1(m^0+m)}+\frac{\lambda_{01}^2}{\mu_1(J^0+J)};\quad \nu_2=\frac{\lambda_2^2}{\mu_2(m^0+m)}+\frac{\lambda_{02}^2}{\mu_2(J^0+J)}; \quad (3.50)$$

$$\nu_3=\frac{(\lambda_{01}\lambda_2-\lambda_{02}\lambda_1)^2}{\mu_1\mu_2(m^0+m)(J^0+J)};\quad \nu_{1\zeta}=\frac{\lambda_{01}\lambda_1}{\mu_1(J^0+J)(m^0+m)}\left(\frac{P}{\sigma^2}\right);$$

$$\nu_{2\zeta}=\frac{\lambda_{02}\lambda_2}{\mu_2(J^0+J)(m^0+m)}\left(\frac{P}{\sigma^2}\right).$$

On the basis of the properties of the system (3.47) the coefficients of equation (3.49) are positive. Here

$$0<\nu_1<1;\ 0<\nu_2<1;\ 0\leqslant\nu_3<1.$$

The criterion of dynamic stability of the space vehicle in the given case has the form

$$D=4[1+\nu_{1\zeta}+\nu_{2\zeta}][1-(\nu_1+\nu_2)+\nu_3]-[2-(\nu_1+\nu_2)+(\nu_{1\zeta}+\nu_{2\zeta})]^2<0. \tag{3.51}$$

Let us introduce the following dimensionless parameters:

$$Z_l=\frac{1}{l}\ \frac{\lambda_{0l}}{\lambda_l};\quad k=\frac{\lambda_2^2\mu_1}{\mu_2\lambda_1^2};\quad \gamma=\frac{\lambda_1^2}{\mu_1(m^0+m)};$$
$$\zeta=\frac{R_{01}}{2l\bar{\sigma}^2};\quad l^2=\frac{J^0+J}{m^0+m};\quad \bar{\sigma}^2=\frac{R_{01}}{P}(m^0+m)\sigma^2; \tag{3.52}$$

$$k>0;\ 0<\gamma<1;\ 0<\zeta\gamma<1. \tag{3.53}$$

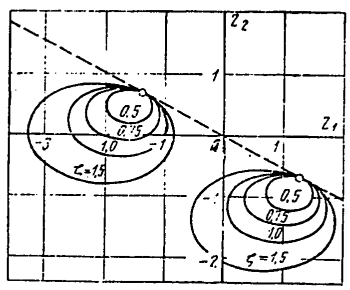

Figure 3.11. Boundaries of the regions of dynamic instability of the space vehicle of the classical system in the plane of the parameters Z_1, Z_2: $0{,}5\leqslant\zeta\leqslant1{,}5$; $k=2{,}0$; $\gamma=0{,}1$

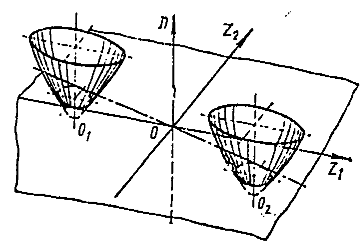

Figure 3.12. Discriminant D of the characteristic equation $\Phi(p)=0$ as a function of the parameters Z_1, Z_2

Using (3.52), let us reduce the relations (3.53) and the stability criterion (3.51) to the following, respectively

$$\nu_1=\gamma(1+Z_1^2);\ \nu_2=\gamma k(1+Z_2^2);\ \nu_3=\gamma^2k(Z_2-Z_1)^2;$$
$$\nu_{1\zeta}=2\zeta\gamma Z_1;\ \nu_{2\zeta}=2\zeta\gamma kZ_2; \tag{3.54}$$
$$D=4k(Z_2-Z_1)^2[1+2\zeta\gamma(Z_1+kZ_2)]-[Z_1^2+kZ_2^2+2\zeta(Z_1+kZ_2)]^2<0.$$

In the space of the parameters Z_1, Z_2, γ, k and ζ the equation D=0 finds the surface which separates the regions of stability and the regions of instability of the investigated system (Fig 3.11). From the structure of the inequality (3.54) it follows that the equation D=0 is conveniently considered as the equation of the plane curve in the variables Z_1, Z_2 considering the remaining variables as parameters of this curve. Then by the region of stability of the system (3.47) in the absence of dissipation we mean the region (3.54) bounded by the curve D=0 in the plane (Z_1, Z_2).

The form of the curves D=0 for various values of the parameter ζ and fixed values of the parameters k, γ (k=2.0; ζ=0.1) is presented in Fig 3.12. The regions of instability of the system for each of the values of ζ are included inside the corresponding closed curves close, as will be seen from what follows, to ellipses. The numerical analysis shows that the regions of instability have similar form for all values of the parameters k, γ, satisfying the expressions (3.53) (Fig 3.12, 3.13).

Let us explain the geometric and physical meaning of the parameters Z_1, Z_2 in the simplest case of a body with two cylindrical cavities of identical radius R.

Here we propose that the depth of the liquid h_i (i=1,2) in both compartments is sufficiently great: $h_i \gtrsim 2R$. The hydrodynamic coefficients of the system of equations (3.1) in this case have the form

$$\lambda_1=\frac{\pi}{\xi^2}\varrho_1R^3;\quad \lambda_2=\frac{\pi}{\xi^2}\varrho_2R^3;\quad \sigma_1^2=\sigma_2^2=\frac{P}{(m^0+m)R}\xi;$$
$$\mu_1=\frac{\pi(\xi^2-1)}{2\xi^2}\varrho_1R^3;\quad \mu_2=\frac{\pi(\xi^2-1)}{2\xi^2}\varrho_2R^3; \tag{3.55}$$
$$\lambda_{01}=-\frac{\pi}{\xi^2}\left[x_1+h_1-\frac{R}{\xi}\right]\varrho_1R;\quad \lambda_{02}=-\frac{\pi}{\xi^2}\left[x_2+h_2-\frac{R}{\xi}\right]\varrho_2R^3.$$

Here ξ=const=1.841; x_i are the coordinates of the bottoms of the compartments in the coordinate system connected with the space vehicle; ρ_i is the fuel density in the compartments.

Consdiering the equations (3.55), the parameters (3.52) assume the form

$$Z_i=-\frac{1}{l}(x_i+h_i-0{,}54R);\quad \gamma=1{,}43\frac{\varrho_1R^3}{m^0+m};\quad k=\frac{\varrho_2}{\varrho_1};\quad \zeta=0{,}27R/l.$$

As is obvious, the parameters Z_i (i=1,2) are close to the dimensionless coordinates of the free surfaces of the fuel in the compartments, the parameter k characterizes the relative density of the liquids, the parameter γ defines the degree of physical connectedness of the wave movements of the liquid with displacement of the center of masses of the space vehicle and, finally the parameter ζ characterizes the relative elongation of the space vehicle in the direction of the longitudinal axis.

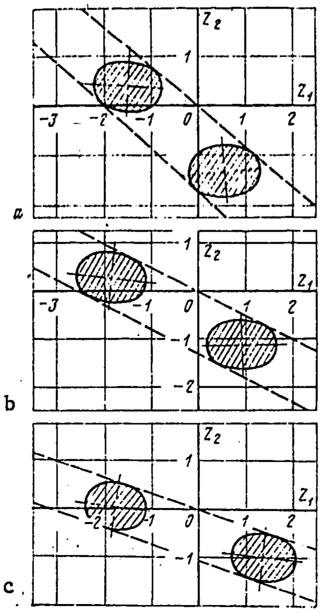

Figure 3.13. Nature of the effect of various partial frequencies of oscillations of the fuel in the combustible fuel component and oxidizing agent tanks ($\beta=-1+\sigma_2^2/\sigma_1^2$) on the regions of dynamic instability of a space vehicle

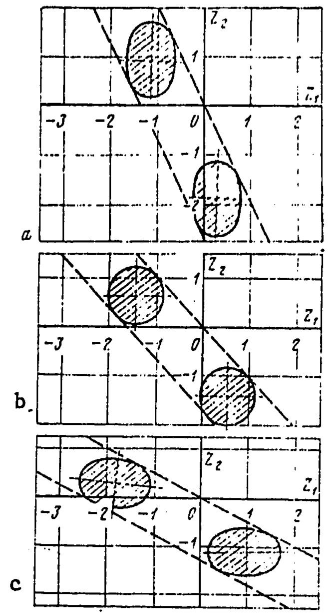

Figure 3.14. Nature of the effect of the densities of the fuel components on the region of dynamic instability of the space vehicle
a -- parameter k=0.5; b -- parameter k=1; c -- parameter k=2

In the more general case $\sigma_1 \neq \sigma_2$ it is necessary to add another parameter $\beta=(-1+\sigma_2^2/\sigma_1^2)$ characterizing the development of the partial frequencies of the liquid in both cavities to the set of dimensionless parameters (3.52). The stability criterion (3.51) assumes the form

$$D=-(\nu_1+\nu_2+\nu_{1\zeta}+\nu_{2\zeta})^2+4\nu_3(1+\nu_{1\zeta}+\nu_{2\zeta})+ \\ +\beta k\gamma\{2(1-\nu_1)[2-(\nu_1+\nu_2)+\nu_{1\zeta}+\nu_{2\zeta}]+ \\ +\beta k\gamma(1-\nu_{1\zeta})[1-(\nu_1+\nu_2)+\nu_3]<0, \qquad (3.56)$$

where ν_2 (r=1,2,3) and $\nu_{\ell\zeta}$ (ℓ=1,2) are defined by the expressions (3.50).

As numerical analysis shows (see Figures 3.13, 3.14), the set of curves D=0 bounding the regions of instability, has the same form as in Fig 3.11.

Structural Stability Criterion

The characteristic equation of the closed system (3.47)-(3.48) is representable in the form

$$p^2(a_0p^4+a_1p^2+a_2)+L(p)(b_0p^4+b_1p^2+b^2)=0. \qquad (3.57)$$

Let us denote

$$\Phi_0(p^2)=p^2(a_0p^4+a_1p^2+a_2);\ \Phi_k(p^2)=b_0p^4+b_1p^2+b_2;$$
$$a_0=1-(\nu_1+\nu_2)+\nu_3;\ a_1=[2-(\nu_1+\nu_2)+(\nu_{1c}+\nu_{2c})]\sigma^2;\ a_2=(1+\nu_3)\sigma^4;$$
$$b_0=b_{\psi u}(-1+\nu_4);\ b_1=\sigma^2 b_{\psi u}(-2+\nu_4);\ b_2=-\sigma^4 b_{\psi u}, \qquad (3.58)$$

where in addition to (3.54) the following is denoted

$$\nu_4=\sum_{i=1}^{2}\left[\frac{\lambda_i^2}{\mu_i(m^0+m)}+\frac{\lambda_i\lambda_{0i}b_{zu}}{\mu_i b_{\psi u}(m^0+m)}\right].$$

The criterion of structural stazbility of the system as the criterion of alternatability of the roots of the equations $\Phi_0=0$, $\Phi_k=0$ reduces to one inequality:

$$\Delta_4=(a_1b_2-a_2b_1)(a_0b_1-a_1b_0)-(a_0b_2-b_0a_2)^2>0$$

or considering the equations (3.58):

$$(\nu_1+\nu_2)\nu_4-\nu_4^2-\nu_3+(\nu_{1c}+\nu_{2c})\nu_4-(\nu_{1c}+\nu_{2c})\nu_4^2>0. \qquad (3.59)$$

As the characteristic parameters for the given system let us select the set of parameters (3.52), adding the parameter $\beta=(-1+\sigma_2^2/\sigma_1^2)$ to it.

The regions of structural stability of the spacecraft will be constructed in the plane of the variables Z_1, Z_2, and in this case the remaining variables acquire the meaning of the parameters of the boundaries of these regions. By definition, we have $\zeta>0$; $-\infty<c<\infty$; $c=Z_0^{-1}$.

The regions of application of the other parameters follow from the condition of positive determinacy of the quadratic form corresponding to the kinetic energy of the investigated mechanical system:

$$|Z_1|<\frac{1}{\sqrt{\gamma}};\quad |Z_2|<\frac{1}{\sqrt{k\gamma}};\quad 0<k<\frac{1-\gamma}{\gamma}. \qquad (3.60)$$

The criterion of structural stability of the system (3.59) expressed in terms of the dimensionless parameters Z_1, Z_2, c, γ, k, β, ζ has the form

$$\Psi=(Z_2-\eta_1 Z_1)(Z_2-\eta_1 Z_1)(a_1 Z_1^3+a_2 Z_1^2 Z_2+a_3 Z_1 Z_2^2+a_4 Z_2^3+$$
$$+a_5 Z_1^2+a_6 Z_1 Z_2+a_7 Z_2^2+a_8 Z_1+a_9 Z_2+a_{10})<0, \qquad (3.61)$$

where

$$\eta_{1,2}=\frac{1}{2}[(1-k_1+\beta)\pm\sqrt{(1-k_1+\beta)^2+4k_1}];$$

$$a_1=k_1 c+2k_1^2 c\gamma_1\zeta;\ a_2=-k_1 c+2k_1 c^2\gamma_1\zeta(2-k_1);\ a_3=c+2c^2\gamma_1\zeta(1-2k_1);$$
$$a_4=-c+2c^2\gamma_1\zeta;\ a_5=k_1(c^2-1)+2k_1 c\zeta-4k_1 c\gamma_1\zeta(1+k_1); \qquad (3.62)$$
$$a_6=-\beta c^2+(c^2-1)(1-k_1+\beta)+2\zeta(c-2c\gamma-$$
$$-k_1 c-2\beta c\gamma_1+2k_1^2 c\gamma_1-\beta k_1 c\gamma_1);$$
$$a_7=-(c^2-1)+2\zeta(-c+2c\gamma_1+2k_1 c\gamma_1+\beta c\gamma_1);$$
$$a_8=-c(1+k_1+\beta)+2\zeta(1+k_1+\beta)(\gamma_1+k_1\gamma_1+\beta\gamma_1-1)-$$
$$-2\beta k_1\gamma_1\zeta(c^2+2)+2\beta k_1 c\gamma_1\zeta(\beta\gamma_1-1);$$
$$a_9=c(1+k_1-\beta)+2\zeta(1+k_1-\gamma_1-k_1^2\gamma_1-2k_1\gamma_1+$$
$$+\beta k_1\gamma_1-[\beta c^2\gamma_1-\beta\gamma_1-\beta-4\beta c\gamma_1\zeta^2];$$
$$a_{10}=-\beta c^2+2\zeta(-2\beta c+\beta c\gamma_1+\beta k_1 c\gamma_1+\beta^2 c\gamma_1)+$$
$$+4\zeta^2(-\beta+\beta\gamma_1+\beta k_1\gamma_1+\beta^2\gamma_1-\beta^2 k_1\gamma_1^2). \qquad (3.62)$$

The parameters (k, γ) and (k_1, γ_1) are related by the following expressions:

$$k=\frac{1}{k_1(1-\beta\gamma_1)};\quad \gamma=k_1\gamma_1.$$

Fig 3.15 shows the regions of structural stability for the cases $\beta=-1(\sigma_1>\sigma_2)$; $\beta=0(\sigma_1=\sigma_2)$; $\beta=1(\sigma_1<\sigma_2)$ (crosshatched) where Fig 3.15, a,b,c correspond to the value of the parameters c=1, Fig 3.15, d,e,f corresponds to the value of c=0 (the curvilinear boundaries degenerate into rectilinear). For the other dimensionless parameters the following numerical values are assumed:

$$k=2;\quad \gamma=0{,}05;\quad \zeta=0{,}5.$$

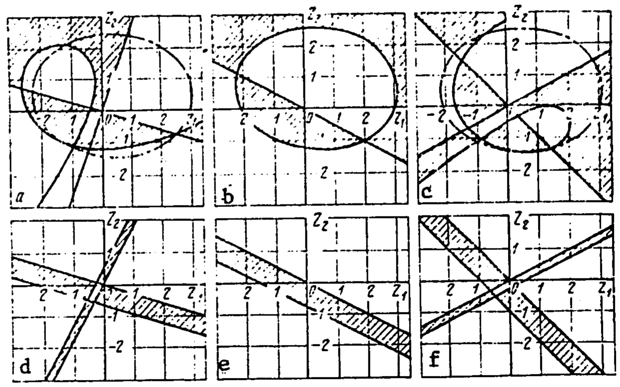

Figure 3.15. Standard regions of structural instability of the space vehicle of the classical system (crosshatched)

Study of the Geometric Configuration of the Regions of Unstabilizability

The functions $\Psi(Z_1, Z_2, c, k, \beta, \gamma, \zeta)$ and $D(Z_1, Z_2, k, \beta, \gamma, \zeta)$ will be represented in the form of the polynomials with respect to powers of the variables Z_1, Z_2:

$$\Psi=(Z_2-\eta_1 Z_1)(Z_2-\eta_2 Z_1)(a_1 Z_1^3+a_2 Z_1^2 Z_2+a_3 Z_1 Z_2^2+a_4 Z_2^3+ \\ +a_5 Z_1^2+a_6 Z_1 Z_2+a_7 Z_2^2+a_8 Z_1+a_9 Z_2+a_{10}), \quad (3.63)$$

where the coefficients a_i are defined by the expressions (3.62) and

$$D=d_1 Z_1^4+d_2 Z_1^3 Z_2+d_3 Z_1^2 Z_2^2+d_4 Z_1 Z_2^3+d_5 Z_2^4+d_6 Z_1^3+ \\ +d_7 Z_1^2 Z_2+d_8 Z_1 Z_2^2+d_9 Z_2^3+d_{10} Z_1^2+d_{11} Z_1 Z_2+d_{12} Z_2^2+ \\ +d_{13} Z_1+d_{14} Z_2+d_{15}, \quad (3.64)$$

where it is denoted

$$d_1=\gamma_1^2(1+\beta k_1\gamma_1);\ d_2=8\zeta k_1\gamma_1^3;\ d_3=2k_1\gamma_1^2(1+\beta k_1\gamma_0);\ d_4=0;\ d_5=k_1^2\gamma_1^2;$$
$$d_6=2\zeta\gamma_1^2(1-\beta k_1\gamma_1);\ d_7=2\zeta k_1\gamma_1^2(1-\beta k_1\gamma_1);\ d_8=8\zeta k_1^2\gamma_1^3+2\zeta k_1\gamma_1^2(1-2\gamma_1);$$
$$d_9=2\zeta k_1^2\gamma_1^2(1-2\gamma_1);\quad d_{10}=\gamma_1^2(1+\beta k_1\gamma_1)[\zeta^2-2\beta k_1(1-\gamma_1)+2(1+k_1)];$$
$$d_{11}=2\zeta^2 k_1\gamma_1^2+8k_1\gamma_1^2(1+8k_1\gamma_1+\beta k_1\gamma_1^2\zeta);\ d_{12}=4k_1\gamma_1[-1+ \\ +(1-\gamma_1)(1+\beta k_1\gamma_1+\beta k_1\gamma_1^2\zeta)]+2k_1\gamma_1^2[1+k_1-\beta k_1(1-\gamma_1)]+\zeta^2 k_1^2\gamma_1^3;$$
$$d_{13}=2\zeta\gamma_1[\gamma_1(1+k_1)+\beta k_1\gamma_1(1-\gamma_1)+2\beta k_1\gamma_1-2k_1\gamma_1^2];\quad d_{14}=2\zeta k_1\gamma_1^2[1+ \\ +k_1-2k_1\gamma_1+\beta k_1\gamma_1(1-\gamma_1)];\ d_{15}=4[-\gamma_1(1+k_1)]+\beta k_1\gamma_1(2-\gamma_1)+ \\ +4[\gamma_1(1+k_1)-k_1\gamma_1^2][1+\beta k_1\gamma_1+\beta k_1\gamma_1^2\zeta-\gamma_1(1+k_1)+ \\ +\beta k_1\gamma_1(1-\gamma_1)]^2.$$

The parameters k, γ, ζ, c, Z_1, Z_2 can assume any values inside the region given by the inequalities (3.60).

As is obvious from the formula (3.63), the regions of structural stability in the plane Z_1, Z_2 in the general case are bounded by the straight lines $Z_2-\eta_1 Z_1=0$, $Z_2-\eta_2 Z_1=0$ and the third-order curve

$$a_1Z_1^3+a_2Z_1^2Z_2+a_3Z_1Z_2^2+a_4Z_2^3+a_5Z_1^2+a_6Z_1Z_2+a_7Z_2^2+ \\ +a_8Z_1+a_9Z_2+a_{10}=0. \qquad (3.65)$$

The type of curve (3.65) is defined by the values of the parameters $\beta=(-1+\sigma_2^2/\sigma_1^2)$ and $c=\frac{x_p-x_O}{l}$. characterizing the ratio of the partial frequencies of the oscillations of the fuel in the compartments and the arm of the controlling force.

For $\beta=0$ ($\sigma_1=\sigma_2$) the equation (3.65) is split into the equation of a straight line and a second-order curve (Fig 3.15, b).

For c=0 (the case of control where the stabilization of the space vehicle is realized by using a pair of forces) the curve defined by the equation (3.65) degenerates into a set of three straight lines (see Fig 3.15, d,e,f). These cases exhaust the possible forms of the curves bounding the regions of structural instability for the investigated classes. Let us consider them successively.

a) General case: $\beta\neq0$; $c\neq0$.

For simplification let us set $\zeta=0$; let us represent the equation (3.65) in the form

$$\Psi(Z_1,\ Z_2)=ck_1Z_1^3-ck_1Z_1^2Z_2+cZ_1Z_2^2-cZ_2^3+k_1(c^2-1)Z_1^2+ \\ +[(c^2-1)(1-k_1)-\beta]Z_1Z_2-(c^2-1)Z_2^2-(1-k_1+\beta)Z_1+ \\ +(1-k_1-\beta)Z_2-\beta c=0. \qquad (3.66)$$

Let us find the asymptote of the curve (3.66), the equations of which will be represented in the form

$$Z_2=a;\ Z_1=b.$$

For determination of the coefficients a and b we have the equations

$$k_1-k_1a+a^2+a^3=0,$$
$$c(-k_1+2a-3a^2)b=-k_1(c^2-1)-[(c^2-1)(1-k_1)-\beta]a-(c^2-1)a^2,$$

the joint solution of which gives only one asymptote

$$Z_2=Z_1-\frac{\beta}{c(1-k_1)}.$$

Let, for example, c>0. Since $k_1>0$, the position of the asymptote is determined by the sign of the parameter β.

The curve (3.66) pertains to the group of defective hyperbolas. The canonical form of its equation has the form

$$xy^2+ey=Ax^3+Bx^2+Cx+D\ (A<0),$$

where
$$e=-\frac{2\lambda}{(1+k_1)^2}\left[\lambda\frac{1-k_1}{2(1+k_1^2)}+n\right];\ \lambda=\frac{\beta}{c};\ n=\frac{c^2+1}{2c};$$

$$A=-k_1;\ B=-4\lambda\frac{k_1}{(1+k_1)^2};\ C=\lambda^2\frac{k_1^2-22k_1+1}{4(1+k_1)^4}+\lambda n\frac{1-k_1}{(1+k_1)^2}+n^2;$$

$$D=\frac{2\lambda}{(1+k_1)^2}\left[\frac{k_1^2-6k_1+1}{4(1+k_1)^4}+\lambda n\frac{1-k_1}{1+k_1^2}+n^2\right].$$

Equation (3.66) is reduced to this form by linear nonsingular transformations (c≠0):

$$Z_1=x+y+\lambda\frac{3-k_1}{2(1+k_1)^2}-\frac{c^2-1}{2c}$$

$$Z_2=-k_1x+y+\lambda\frac{1-3k_1}{2(1+k_1)}-\frac{c^2-1}{2c}.$$

The basic forms of the curves of this group are determined by the form of the roots of the auxiliary equation

$$Ax^4+Bx^3+Cx^2+Dx+\frac{e^2}{4}=0.$$

In the given case

$$x_{1,4}=-\frac{\lambda}{(1+k_1)^2}\mp\sqrt{\frac{\lambda^2}{(1+k_1)^4}+\frac{1}{k_1}\left[2\frac{1-k_1}{1+k_1}+n\right]}.$$

$$x_2=x_3=\frac{\beta}{c(1+k_1)^2}.$$

It is obvious that $x_1<x_2=x_3<x_4$ for all values of the parameters β, k_1, c>0. This conclusion remains valid also for ς≠0. The curve (3.66) bounding the regions of structural stability of the system is presented in Fig 3.15, a, c, where Fig 3.15, a corresponds to the value of β>0, Fig 3.15, c, the value of β<0.

All of the conclusions remain in force also for the values of c<0.

b) The case of β=0 (c≠0).

The system (3.47) is structurally stable if the following inequality is satisfied:

$$(Z_2-Z_1)^2(Z_1+kZ_2)\left\{\left(Z_1+\frac{c^2-1}{2c}+\frac{\zeta}{2}\right)^2+k\left(Z_2+\frac{c^2-1}{2c}+\frac{\zeta}{2}\right)^2-\right.$$
$$\left.-(1+k)\left[\frac{c^2+1}{2c}+\frac{\zeta}{2}\right]^2+c\gamma\zeta\left[Z_1+kZ_2-\frac{1+k}{2}\right]^2\right\}>0. \quad (3,67)$$

The third cofactor in the expression (3.67) in the plane Z_1, Z_2 defines an ellipse which is one of the boundaries of the region of structural stability of the system. The axis of the ellipse makes an angle θ with the coordinate axes Z_1, Z_2. The coordinates of the center (Z_1^0, Z_2^0) and the angle θ are defined by the following formulas:

$$z_1^0=z_2^0=\frac{1-c^2+c\zeta\,[1-2\gamma(1+k)]}{2c\,[1+c\zeta\gamma(1+k)]};$$
$$\theta=\frac{1}{2}\,\mathrm{arctg}\,\frac{2\zeta\gamma ck}{(1-k)\,[1+c\zeta\gamma(1+k)]}.$$

The regions of stability of the system corresponding to the inequality (3.67) are crosshatched in Fig 3.15, b.

c) The case c=0 (β is arbitrary).

This case is realized for control of the system using the pair of forces: setting b_{zu}=0 in the initial system (3.47), we obtain the structural stability criterion of the system in the form

$$(Z_2-\eta_1 Z_1)(Z_2-\eta_2 Z_1)(Z_2-\eta_1 Z_1+b_1)(Z_2-\eta_2 Z_1+b_2)>0,$$

where

$$\eta_{1,2}=\frac{1}{2}\,[(1-k_1+\beta)\pm\sqrt{(1-k_1+\beta)^2+4k_1}];$$
$$b_{1,2}=b_{1,2}\,(k,\ \beta,\ \zeta,\ \gamma).$$

As is obvious, in the given case the regions of structural instability are bounded by a set of straight lines (crosshatched) in Fig 3.15 d, e, f, the position of which with respect to the origin of the coordinates is determined by the sign of the parameter β. The case β=0 (Fig 3.15, e) is the simplest in this case, for the regions of structural instability degenerate into the band

$$Z_2+kZ_1<0;\ Z_2+k_1Z_1+\zeta(1+k_1)[1-\gamma(1+k_1)]>0.$$

Let us return to the analysis of the regions of natural dynamic instability of the system. The boundaries of these regions are defined by the equation

$$D(Z_1,\ Z_2,\ k,\ \beta,\ \zeta,\ \gamma)=0 \quad (3.68)$$

and, as is demonstrated, they are a set of two ovals in the right and left halfplanes of the plane Z_1, Z_2.

For $\zeta\to 0$ and $\beta\to 0$ (a body of relatively large elongation with equal partial oscillation frequencies of the liquid in the cavities) the boundaries of the regions (3.68) are quite precisely approximated by the ellipses

$$(k+\zeta k\sqrt{k}\gamma)Z_1^2+\zeta\gamma\sqrt{k}(1-k)Z_1Z_2+(1-\zeta\gamma\sqrt{k})Z_2^2+ \\ +(2\sqrt{k}+\zeta\sqrt{k})Z_1+(-2\sqrt{k}+\zeta)Z_2+(1+k)=0; \qquad (3.69)$$

$$(k-\zeta k\sqrt{k}\gamma)Z_1^2+\zeta\gamma\sqrt{k}(-1+k)Z_1Z_2+(1+\zeta\gamma\sqrt{k})Z_2+ \\ +(-2\sqrt{k}+\zeta k)Z_1+(2\sqrt{k}+\zeta)Z_2+(1+k)=0 \qquad (3.70)$$

with centers at the points

$$\xi_i=\pm\frac{1}{\sqrt{k}}-\frac{\zeta}{2};\ \varkappa_i=\pm\frac{1}{\sqrt{k}}-\frac{\zeta}{2}.$$

The axes of the ellipses are rotated with respect to Z_1 axis by the angle

$$\theta_i=\frac{1}{2}\,\mathrm{arctg}\,\frac{\zeta\gamma(1+k^2)}{\zeta\gamma(1+k)\mp\sqrt{k}(1-k)},\ i=1,\ 2.$$

Let us investigate the problem of the mutual arrangement of the regions of structural and natural dynamic instability of the space vehicle. Let us consider the simplest case of c=0; β=0 when the boundaries of the regions of structural and natural instability are defined by the equations

$$(Z_1+kZ_2)\{Z_1+kZ_2+\zeta(1+k)[1-\gamma(1+k)]\}=0;$$

$$[Z_1^2+kZ_2^2+(1+k)+\zeta(Z_1+kZ_2)]^2-4k(Z_2-Z_1)^2\times \\ \times[1+\zeta\gamma(Z_1+kZ_2)]=0.$$

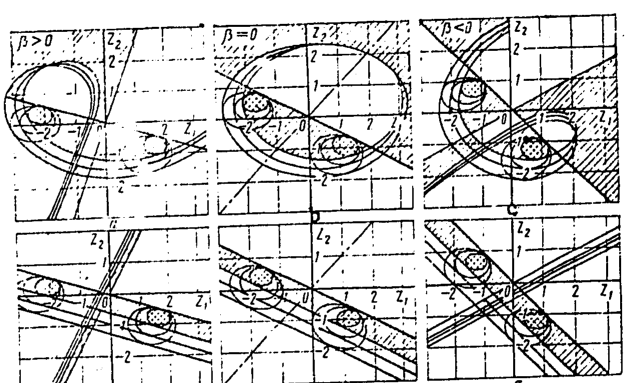

Figure 3.16. Mutual arrangement of the regions of dynamic double crosshatching) and structural instability of the space vehicle with two fuel tanks:
a, b, c -- parameter c>0; d, e, f -- parameter c=0

From the general theory discussed in Chapter 2 it follows that the regions of natural instability are located inside the regions of structural instability.

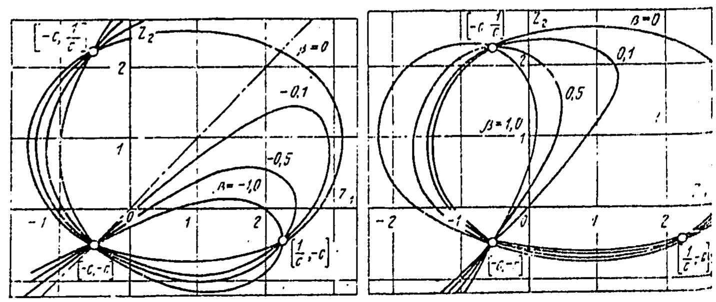

Figure 3.17. Nature of the effect of various oscillation frequencies of the fuel in the tank on the curvilinear boundaries of the regions of structural instability of a space vehicle ($\beta=-1+\sigma_2^2/\sigma_1^2$);
a -- parameter $\beta \leq 0$; b -- parameter $\beta \geq 0$

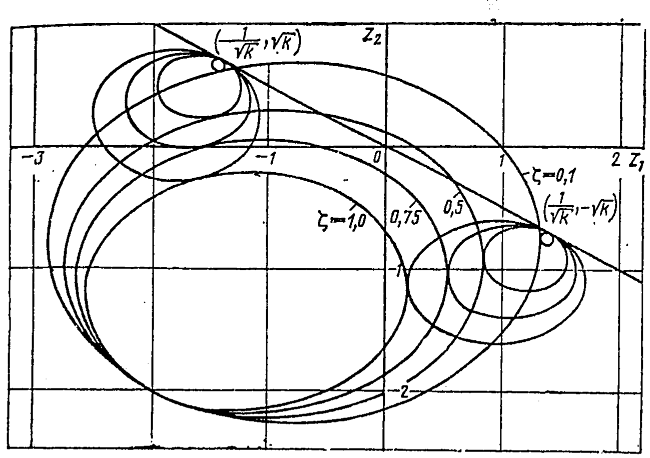

Figure 3.18. Mutual arrangement of the regions of natural and structural stability of a space vehicle for c<0 ("forward" with respect to the metacenter of location of the servoelements):
k=0.5; Z_0=02; γ=0.1

This is, of course, valid also for the investigated system; in the given case there is tangency of the corresponding boundaries at the points

$$\left(-\sqrt{k}, \frac{1}{\sqrt{k}}\right); \left(\sqrt{k}, -\frac{1}{\sqrt{k}}\right);$$
$$[(-\alpha_1+\alpha_2), (-\alpha_1-k\alpha_2)]; [(-\alpha_1, -\alpha_2), (-\alpha_1+k\alpha_2)];$$
$$\alpha_1=\zeta[1-\gamma(1+k)]; \alpha_2=\sqrt{k[1-\zeta\gamma\alpha_1(1+k)]}.$$

The mutual arrangement of the boundaries of the regions $\Psi=0$, $D=0$ is presented in Fig 3.16, e.

In the general case $\beta\neq0$; $c\neq0$ the various versions of the mutual arrangement of the boundaries of the regions $\Psi=0$; $D=0$ are presented in Fig 3.16. Examples of calculations of the regions $\Psi=0$, $D=0$ for standard values of the parameters are presented in Figures 3.17 and 3.18.

3.4. Space Vehicle with Engine on an Elastic Suspension

Statement of the Problem

Let us consider the equations of motion of the space vehicle (3.24) making the following simplifying assumptions:

The space vehicle is controlled by a sustainer engine on a gimbal;

The mass and radius of inertia of the deflected engine are small by comparison with the mass and radius of the inertia of the hull;

The engine has high unit thrust (the ratio of the thrust to its mass);

The frequencies of the elastic oscillations of the hull ω_{qj} in the engine ω_u on the elastic suspension are close in the sense that high connectedness of the corresponding partial systems is insured.

Neglecting the effect of the mobility of the fuel ($\sigma_i \ll \omega_j$) the equations (3.24) of disturbed motion of the space vehicle will be written in the form

$$\ddot{z}+a_{z\psi}\dot{\psi}+\sum_{j=1}^{m}a_{zq_j}q_j+a'_{zu}\ddot{u}+a_{zu}u=0;$$
$$\ddot{\psi}+\sum_{j=1}^{m}a_{\psi q_j}q_j+a'_{\psi u}\ddot{u}+a_{\psi u}=0;$$
$$\ddot{q}_j+\omega_{q_j}^2 q_j+a_{q_j u}\ddot{u}+a_{q_j u}u=0; \qquad (3.71)$$

$$\ddot{u}+\omega_u^2 u+a_{u\psi}(\psi+u)+\sum_{j=1}^{m} a_{uq_j} q_j+a'_{uz}\ddot{z}+\sum_{j=1}^{m} a'_{uq_j}\ddot{q}_j+$$
$$+a_{u\psi}\ddot{\psi}=\omega_u^2 u^0;$$
$$u^0=L(\psi^0);\ \psi^0=\psi-\sum_{j=1}^{m}\eta'_j(x^0)q_j.$$

In the system (3.71), just as before, the following notation is adopted:

z is the displacement in the transverse direction of the point of the hull bounded to the metacenter x_G;

ψ is the angle of rotation of the hull with respect to the transverse axis;

q_j is the generalized coordinate corresponding to the j-th form of elastic vibrations of the hull (j forms are taken into account); in addition, the following notation is used: x^0 is the coordinate of the sensitive element of the stabilization system; u is the angle of rotation of the engine; u^0 is the angle of rotation of the servomechanism.

In the system (3.71) let us retain the equations corresponding to the two oscillatory elements with frequencies of ω_q and ω_u. Omitting the index j, after certain simplifications we find

$$\begin{aligned}&\ddot{\psi}+a_{\psi u}\ddot{u}+a_{\psi u}u=0;\\&\ddot{q}+\omega_q^2 q+a'_{qu}\ddot{u}+a_{qu}u=0;\\&\ddot{u}+\omega_u^2 u+a'_{u\psi}\ddot{\psi}+a'_{uq}\ddot{q}=\omega_u^2 u^0;\\&u^0=L(p)[\psi-\eta'(x^0)q].\end{aligned} \tag{3.72}$$

The characteristic equation of the system (3.72) is represented in the following "standard" form:

$$\Phi_0(p^2)+L(p)\Phi_k(p^2)=0, \tag{3.73}$$

where

$$\Phi_0(p^2)=p^2(a_0p^2+a_1p^2+a_2);$$
$$\Phi_k(p^2)=b_0p^4+b_1p^2+b_2. \tag{3.74}$$

Let us introduce the parameters $\varkappa=1-\frac{\omega_u^2}{\omega_q^2};\ \zeta_q=\frac{a_{\psi u}}{\omega_q^2}$

and we assume that χ and ζ_q are values which are small by comparison with one, which follows from the essence of the investigated problem. The coefficients of the polynomials (3.74) can be represented in the form

$$a_0=1-(a'_{\psi u}a_{u\psi}+a'_{qu}a'_{uq});\ b_0=\omega_u^2(a'_{\psi u}-a'_{\psi q}a_{qu});$$
$$a_1=\omega_q^2\left(2-\varkappa-a'_{\psi u}a'_{u\psi}-a'_{u\psi}+\frac{a_{qu}}{a_{\psi u}}a'_{uq}\right);\ b_1=\omega_u^2\omega_q^2\times$$
$$\times\left[a'_{\psi u}+\zeta_q\left(1-\frac{a_{qu}}{a_{\psi u}}a'_{\psi q}\right)\right]; \quad (3.75)$$
$$a_2=\omega_q^4(1-\varkappa-a'_{u\psi}\zeta_q);\ b_2=\omega_u^2\omega_q^2\zeta_q.$$

The system (3.72) is naturally dynamically stable if

$$D=a_1^2-4a_0a_2>0, \quad (3.76)$$

and structurally stable if

$$\Psi=-(b_0a_2-a_0b_2)^2+(b_1a_2-a_1b_2)(b_0a_1-a_0b_1)>0. \quad (3.77)$$

The problem of investigating the system (3.72) thus reduces to analysis of the structural properties of the system based on inequalities (3.76), (3.77) and the dependence of their lefthand sides on the characteristic parameters of the space vehicle.

Let us introduce several groups of parameters in the investigated system.

The parameters characterizing the elastic properties of the body:

$$Z_1=\frac{x_G-x_Q}{l\sqrt{\mu}}\gtrless 0;\ Z_2=\eta(x_Q)\sqrt{\frac{m}{a\mu}}\gtrless 0; \quad (3.78)$$

The parameters characterizing the position of the axis of suspension of the engines:

$$y_0=\frac{x_Q-x_D}{l^*}\gtrless 0;\ y_2=\frac{x_P-x_Q}{l^*}\gtrless 0; \quad (3.79)$$

The parameters directly characterizing the dynamic interaction between the hull and the engines:

$$y_1=\frac{l^*\eta'(x_Q)}{\eta(x_Q)};\ \alpha=\frac{1}{2\omega^{*2}}(\omega_u^2+\omega_q^2);\ \beta=\frac{l^*}{l\sqrt{\mu}};\ \gamma=\frac{2m^*}{m};$$
$$\beta\ll 1;\ \gamma\ll 1; \quad (3.80)$$

The auxiliary parameters: the dimensionless mass of the system μ reduced to the initial mass, the characteristic frequency, the unit thrust and the radii of inertia of the engine and the body with respect to the axis of suspension and the metacenter, respectively:

$$\omega^*=\sqrt{\frac{P^*g}{l^*}};\ \mu=\frac{m}{m_0};\ l^*=\sqrt{\frac{J^*}{m^*}};\ l=\sqrt{\frac{J}{m}}. \quad (3.81)$$

The characteristic points x which figure in expressions (3.78)-(3.81) are as follows: D -- center of mass of the engine; Q -- axis of the hinged suspension; P -- plane of the bearing frame of the form on which the engines are mounted; G -- metacenter of the body-liquid system, $\eta(x)$, $\eta'(x)$ -- form and derivative of the form of the elastic vibrations of the hull; a -- generalized mass corresponding to this form.

In addition to the enumerated parameters not connected with the control operator, let us introduce the parameter

$$\varepsilon = \frac{\eta'(x^0)\, l}{\sqrt{\mu}\,\eta(x_Q)}, \qquad (3.82)$$

characterizing the sensitivity of the elastic system to the actions of this operator.

Study of the Regions of Dynamic Instability

The equation of the boundary of the region of dynamic instability can be reduced to the following form considering expressions (3.78)-(3.80):

$$[Z_0Z_2^2 - Z_1(y_0Z_1+\beta) + \alpha(y_0Z_1+\beta)^2]^2 = \frac{4\alpha Z_0Z_2^2}{\gamma}(1-\alpha Z_0)\left[1-\frac{\gamma}{\alpha}Z_1(y_0Z_1+\beta)\right]. \qquad (3.83)$$

In practice it is usually possible to consider $\beta \approx 0$, $y_2 \approx 0$, $Z_0 \approx y_0 - y_1$.

The equation (3.83) has real solutions Z_1, Z_2 only with a positive righthand side:

$$Z_0(1-\alpha Z_0) > 0.$$

If we exclude the degenerate case of $Z_0=0$ which is of no interest, it is possible to define the upper critical value of the parameter $\alpha=\alpha^0=1/Z_0$.

For $\alpha>\alpha^0$ the instability is impossible; for $\alpha<\alpha^0$ it is a region of instability, the boundaries of which can be found from the equation (3.78).

Let us introduce the variables ζ_1, ζ_2 and parameters ζ, δ:

$$\zeta_1 = y_0Z_1^2;\ \zeta = \alpha_0 y_0 \gtrless 0; \qquad \zeta_2 = Z_0Z_2^2;\ \delta = \alpha y_1 \gtrless 0. \qquad (3.84)$$

In new variables we have

$$(1+\zeta)^2\zeta_1^2 + 2(2-\zeta+2\delta)\zeta_1\zeta_2 + \zeta_2^2 - \frac{4\alpha}{\gamma}(1-\zeta+\delta)\zeta_2 = 0, \qquad (3.85)$$

where these variables are defined in the following quadrants of the plane (ζ_1, ζ_2)

$$\zeta_1 > 0,\ \zeta_2 > 0 \text{ for } \zeta > 0; \qquad (3.86)$$
$$\zeta_1 < 0,\ \zeta_2 > 0 \text{ for } \zeta < 0.$$

The equation (3.85) defines the second-order curve passing through the origin of the coordinates.

In the sense of the problem $\alpha^0 - \alpha > 0$; $\zeta \neq 0$, so that the following inequalities are valid:

$$0 < \zeta - \delta < 1 \text{ for } \delta \leqslant 0;\ 1 - \zeta > 0 \text{ for } \delta \leqslant 0. \qquad (3.87)$$

Then, on the basis of the structural arguments the following estimates are valid

$$|y_0| \ll 1 \text{ for } y_0 < 0;\ |y_1| \ll 1.$$

Calculating the invariants of the curve (3.85)

$$J_1 = 1 + (1-\zeta)^2;\ J_2 = -4\delta(1-\zeta+\delta);\ J_3 = -\frac{4\alpha^2}{\gamma^2}(1-\zeta^2)(1-\zeta+\delta)^2, \qquad (3.88)$$

considering (3.86)-(3.88) we find that depending on the magnitudes and signs of the parameters ζ and δ the following cases are possible:

a) $\delta < 0;\ \delta < \zeta < 1 + \delta.$

Equation (3.83) defines the ellipse with the center at the point

$$\zeta_1^0 = \frac{\alpha(1-\zeta+2\delta)}{2\gamma\delta} < 0;\ \zeta_2^0 = -\frac{\alpha(1-\zeta)^2}{2\gamma\delta^2} > 0, \qquad (3.89)$$

the axes of which are rotated by the angle

$$\varphi_1 = -\frac{\pi}{2} + \frac{1}{2}\operatorname{arctg}\frac{2(1-\zeta+2\delta)}{(1-\zeta)^2-1}, \text{ if } \delta < \zeta < 0;$$
$$\varphi_2 = -\frac{1}{2}\operatorname{arctg}\frac{2(1-\zeta+2\delta)}{(1-\zeta)^2-1}, \text{ if } 0 < \zeta < 1 + 2\delta.$$

In Fig 3.19, a, b the regions D<0 are double crosshatched. However, the regions located in the first quadrant ((for $\delta<\zeta<0$) or in the second quadrant (for $0<\zeta<1+2\delta$) have physical meaning.

b) $\delta = 0,\ 0 < \zeta < 1.$

The equation (3.85) defines a parabola with the axis rotated by the angle $\phi = -\operatorname{arctg}(1-\zeta)$ with respect to the $0\zeta_1$ axis and the focal parameter

$$p = -\frac{2\alpha(1-\zeta)^2}{\gamma[(1-\zeta)^2+1]\sqrt{(1-\zeta)^2+1}}.$$

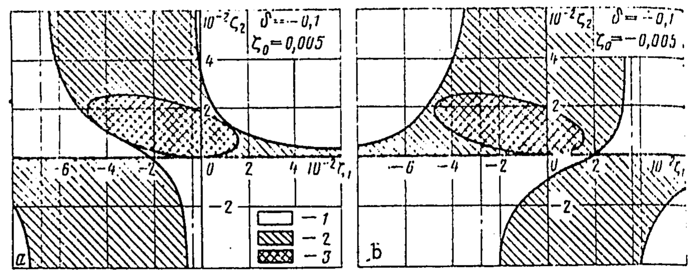

Figure 3.19. Regions of stabilizability (1), unstabilizability (2) and natural dynamic instability (3) of a space vehicle with the engine on an elastic suspension for δ=-0.1

The regions D<0 are represented in the given case in Fig 3.20, a and b (double crosshatched).

The regions of stability (instability) located in the first quadrant have physical meaning.

$$\text{c) } \delta>0,\ \delta<\zeta<1+\delta,\ \zeta\neq 1.$$

The equation (3.15) defines the hyperbola with its center at the point

$$\zeta_1^0=\frac{\alpha(1-\zeta+2\delta)}{2\gamma\delta}>0;\ \zeta_2^0=-\frac{\alpha(1-\zeta)^2}{2\gamma\delta}$$

with the axes rotated by the angle

$$\varphi=-\frac{1}{2}\operatorname{arctg}\frac{2(1-\zeta+2\delta)}{1-(1-\delta)^2}.$$

The regions D<0 are presented in Fig 3.21, a and b (double crosshatched). The regions in the first quadrant have physical meaning.

$$\text{d) } \delta>0;\ \zeta=1.$$

The equation (3.83) assumes the form

$$\zeta_2\left(\zeta_2+4\delta\zeta_1-\frac{4\alpha\delta}{\gamma}\right)=0$$

and, as is obvious, defines the pair of straight lines:

$$\zeta_2=0;\ \zeta_2=-4\delta\zeta_1+\frac{4\alpha\delta}{\gamma}.$$

The straight lines (3.89) coincide with the asymptotes (for $\zeta \to 1$) of the hyperbola, for in this case

$$\zeta_1^0 \to \alpha\gamma^{-1}; \ \zeta_2^0 \to 0; \ \mathrm{tg}\, 2\varphi \to -4\delta.$$

The regions D<0 are presented in Fig 3.21; only the regions of the first quadrant have physical meaning.

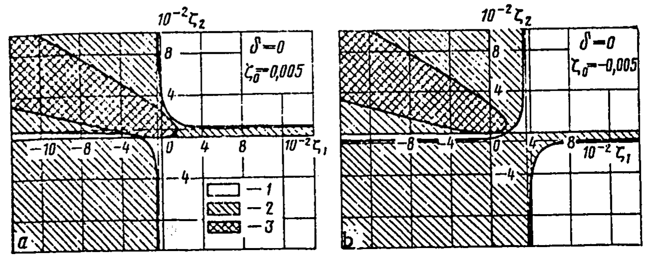

Figure 3.20. Regions of stabilizability (1), unstabilizability (2) and natural dynamic instability (3) of the space vehicle with engine on elastic mounting for $\delta=0$

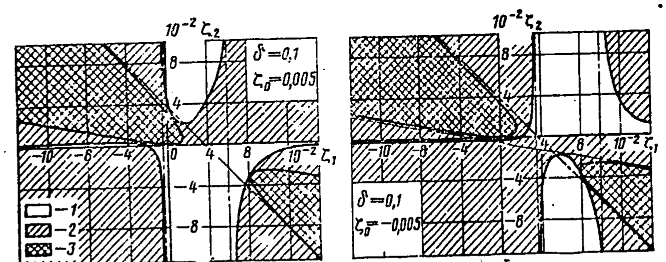

Figure 3.21. Regions of stabilizability (1), unstabilizability (2) and natural dynamic instability (3) of the space vehicle with engine on elastic suspension for $\delta=0.1$

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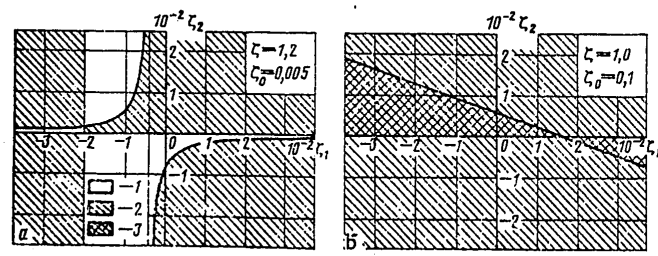
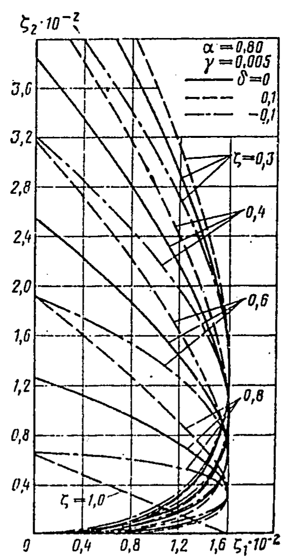


Figure 3.22. Regions of stabilizability (1), unstabilizability (2) and natural dynamic instability (3) of a space vehicle with engine on elastic mounting: ξ, ξ_0 -- parameters for δ=0.1.



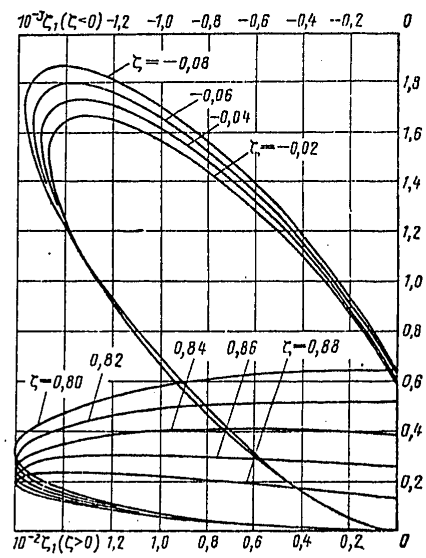


Figure 3.23. Nature of the effect of the parameters δ, ζ on the region of dynamic instability of the system made up of the space vehicle and the engine on an elastic suspension

Figure 3.24. Nature of the effect of the parameter ζ on the regions of dynamic instability of the system made up of the space vehicle and the engine on an elastic suspension: α=0.80; δ=-0.10; γ=0.005.

Additional elements of the boundaries of the regions of natural dynamic instability of the system (3.72) are presented in Fig 3.22, a and b for the values of the parameters ζ and δ.

The coordinates of the apexes of the corresponding curves

$$\zeta_1^0=\alpha\gamma^{-1},\ \zeta_2^0=\frac{\alpha(1-\zeta)}{\gamma}$$

do not depend on the parameter δ, which is illustrated by Figures 3.23 and 3.24.

Thus, the equations of the conical cross sections defining the boundaries of the regions of natural dynamic stabilit of the object of control in the plane of dimensionless parameters ζ and δ were obtained. The parameter δ characterizes the topologic properties of these boundaries, and the parameters $\zeta \neq 1$, the metric properties of the boundaries.

The straight line $\zeta_1{}^0=\alpha\gamma^{-1}$ is the envelope for these boundaries of the regions of instability with fixed value of $\zeta_1{}^0$ for any combinations of parameters ζ, δ.

Study of the Regions of Structural Instability

The equation of the boundaries of the regions of structural instability (3.77) considering the relations (3.78)-(3.79) has the form

$$\begin{aligned}\Psi=&-\gamma Z_0 Z_2^2\left(\frac{1}{\alpha}-Z_0\right)\left(A-\frac{1}{\alpha}Z_1\right)^2+\frac{\gamma^2}{\alpha^2}Z_0^2 Z_1 Z_2^4\times\\&\times\left(\frac{1}{\alpha}-Z_0\right)(-A+Z_0Z_1)+\frac{\varepsilon\gamma}{\alpha}Z_2^2\left(\frac{1}{\alpha}-Z_0\right)\Big\{-\alpha A\times\\&\times\left(A-\frac{1}{\alpha}Z_1\right)^2+Z_0Z_2^2\left[A-Z_0Z_1+Z_1\left(\frac{1}{\alpha}-Z_0\right)\right]-\varepsilon^2 Z_2^4\times\\&\times\left(\frac{1}{\alpha}-Z_0\right)^2+\frac{2\varepsilon\gamma^2}{\alpha^2}Z_0Z_1Z_2^4A\left(\frac{1}{\alpha}-Z_0\right)(-A+Z_0Z_1)+\\&+\frac{\varepsilon^2\gamma}{\alpha}Z_2^4A\left(\frac{1}{\alpha}-Z_0\right)\left[A-Z_0Z_1+Z_1\left(\frac{1}{\alpha}-Z_0\right)\right]+\\&+\frac{\varepsilon^2\gamma^2}{\alpha^2}Z_1Z_2^4A^2\left(\frac{1}{\alpha}-Z_0\right)(-A+Z_0Z_1)=0.\end{aligned}\tag{3.90}$$

As the characteristic parameters let us introduce

$$\zeta_1=y_0Z_1^2;\ \zeta_2=Z_0Z_1^2;\ \zeta=\alpha y_0;\ \delta=\alpha y_1;\ \zeta_0=\frac{\varepsilon}{Z_0Z_1}.\tag{3.91}$$

In accordance with what has been stated above setting β=0, it is possible to reduce equation (3.90) considering (3.91) to the form

$$\frac{1-\alpha Z_0}{\alpha^3} Z_1^2\zeta_2\left\{-\frac{\zeta_0^2\delta}{\alpha}\zeta_1^2\zeta_2+\left[\frac{\zeta_0^2}{\gamma}(1-\zeta+2\delta)-\frac{2\zeta_0\delta}{\alpha}\right]\zeta_1\zeta_2-\right.$$
$$-\frac{\zeta_0}{\gamma}(1-\zeta)^2-\left[\frac{\delta}{\alpha}+\frac{\alpha\zeta_0}{\gamma^2}(1-\zeta+\delta)-\frac{\zeta_0}{\gamma}(1-\zeta+2\delta)\right]\zeta_2-$$
$$\left.-\frac{(1-\zeta)^2}{\gamma}\right\}=0. \qquad (3.92)$$

As is obvious from the expression (3.92), the regions of structural instability are bounded by a straight line

$$\zeta_2=0 \qquad (3.93)$$

and the set of curves defined by the expression in braces which after simplification assumes the form

$$\zeta_2=\varkappa\frac{1+\zeta_0\zeta_1}{(\zeta_0\zeta_1+1-c)\left[\zeta_0\zeta_1+1-(1-\zeta+\delta)\frac{c}{\delta}\right]}. \qquad (3.94)$$

Let us consider successively the same cases as above.

a) $\delta<0;\ \delta<\zeta<1+\delta.$

The regions of structural instability are presented in Fig 3.19, a, b (crosshatched) correspondingly for the cases

$$\zeta_0>0;\ \zeta_0<0.$$

b) $\delta=0;\ \delta<\zeta<1.$

Setting $\delta=0$ in equation (3.92) we find that the regions of structural instability are bounded by the straight line (3.93) and the hyperbola

$$\zeta_2=\frac{1-\zeta}{\zeta_0}\,\frac{\zeta_0\zeta_1+1}{\zeta_0\zeta_1+1-c} \qquad (3.95)$$

with the center at the point

$$\zeta_1^0=-\frac{1-c}{\zeta_0},\ \zeta_2^0=\frac{1-\zeta}{\zeta_0}$$

[crosshatched in Fig 3.20, a ($\zeta_0>0$) and 3.20, b ($\zeta_0<0$)].

The common points of the boundaries of the natural and structural stability of the given case satisfy the equation

$$\zeta_1^*=\frac{c}{2\zeta_0}\left[1+\sqrt{1+\frac{4}{c}}\right],\ \zeta_2^*=\frac{1-\zeta}{\zeta_0}\,\frac{\zeta_0\zeta_1^*+1}{\zeta_0\zeta_1^*+1-c};$$
$$\zeta_1^{**}=\frac{c}{2\zeta_0}\left[1-\sqrt{1+\frac{4}{c}}\right];\ \zeta_2^{**}=\frac{1-\zeta}{\zeta_0}\,\frac{\zeta_0\zeta_1^*+1}{\zeta_0\zeta_1^{**}+1-c}. \qquad (3.96)$$

c) $\delta>0;\ \delta<\zeta<1+\delta.$

The region of structural instability of the system calculated by the boundary equations (3.93), (3.94) are presented in Fig 3.21, a ($\zeta_0>0$) and 3.21, b ($\zeta_0<0$).

d) $\delta>0$; $\zeta=1$.

Setting $\zeta=1$ in the equation (3.92), we find that the criterion of structural stability of the system (3.92) assumes the form

$$\Psi=-\frac{1-\alpha Z_0}{\alpha^3} Z_1^2\zeta_1^2[\zeta_0\zeta_1+1-c]^2>0. \qquad (3.97)$$

For $1-\alpha Z_0>0$ (the conditions of the possibility of natural dynamic instability), as is obvious from equations (3.96), the region of structural instability occupies the entire plane (ζ_1, ζ_2).

For $1-\alpha Z_0<0$ the naturally dynamic stable system (3.92) is also structurally stable.

In conclusion let us consider another case (degenerate) where $\zeta=1+\delta$ and $D=(\delta\zeta_1+\zeta_2)^2\geqslant 0$ (the system is naturally dynamically stable or, perhaps, it is on the stability boundary). In this case, we have

$$\Psi(\zeta_0\zeta_1+1)\left[\zeta_2(\zeta_0\zeta_1+1-c)+\frac{\delta\alpha}{\gamma}\right]>0. \qquad (3.98)$$

The regions of structural instability $\Psi\leqslant 0$ are presented in Fig 3.22 (crosshatched).

Now if we construct the line of the state of the system on the plane ζ_1, ζ_2, each point of which corresponds to some dimensionless parameter τ, $0<\tau\leqslant 1$, the position of this line with respect to the boundaries defines the nature of the stability (instability) of the system for each value of τ.

3.5. Other Examples of Investigation of the Stabilizability of a Space Vehicle

Mutual Effect of the Oscillations of the Fuel in the Compartments and the Elastic Oscillations of the Hull of the Space Vehicle

Let us consider the system of equations (3.24), retaining two additional oscillatory elements in it corresponding to the oscillations of the fuel in any compartment (s) and one of the forms of the oscillations of the hull (a):

$$\ddot{\zeta}+a_{\zeta s}\ddot{s}=-a_{\zeta u}u;$$
$$\ddot{\psi}_0+a_{\psi s}\ddot{s}+a_{\psi q}=-a_{\psi u}u;$$
$$\ddot{q}+\omega_q^2 q+a_{qs}\ddot{s}=-a_{qu}u;$$
$$\ddot{s}+\omega_s^2 s+a_{s\zeta}\ddot{\zeta}+a_{s\psi}\ddot{\psi}_0+a'_{sq}\ddot{q}=0;$$
$$u=L(\psi_0),$$

where

$$a_{\psi q}=\eta'(x_0);\quad a'_{sq}=a_{sq}+a_{\zeta\psi}a_{\psi q}.$$

Let us use the following expressions for the coefficients of the system

$$a_{\zeta s}=-\frac{\varrho R^3}{m^0+m};\quad a_{s\zeta}=-\frac{\lambda}{\mu};\quad a_{qu}=-\frac{2P}{a}\eta_p;$$
$$a_{\psi s}=-\frac{\varrho R^3}{a}\frac{\lambda}{\mu}\left\{\eta_a-\left[\frac{\lambda_{01}}{\lambda}R+x_G-x_a\right]\right\};\quad a_{\zeta u}=-\frac{2P}{m^0+m};$$
$$a_{sq}=-\frac{\lambda}{\mu}\left\{\eta_a-\left[\frac{\lambda_{01}}{\lambda}R+x_G-x_a\right]\right\};\quad a_{\psi u}=-\frac{2P}{J^0+J}(x_G-x_p),$$

where, in addition to the adopted notation, η_a is the form of the elastic line of the space vehicle at the point of attachment of the suspended fuel compartment characterized by the coordinate x_a;

$$a=\int_0^l \eta_p^2\mu(x)\,dx$$

is the apparent mass corresponding to the investigated form of elastic vibrations; $\mu(x)$ is the "running" mass of the hull.

In system (3.97), we also neglected the term $a_{\zeta\psi}$ which in the given case plays a secondary role. The characteristic equation of the system (3.97) will be written in standard form:

$$\Phi_0(p^2)+L(p)\Phi_\kappa(p^2)=0,$$

where

$$\Phi_0(p^2)=p^2(a_0p^4+a_1p^2+a_2);\quad \Phi_\kappa(p^2)=b_0p^4+b_1p^2+b_2;$$
$$a_0=1-(a_{\zeta s}a_{s\zeta}+a_{\psi s}a_{s\psi}+a_{qs}a_{sq});$$
$$a_1=\omega_q^2\left[2-(a_{s\zeta}a_{\zeta s}+a_{s\psi}a_{\psi s})-\left(1-\frac{\omega_s^2}{\omega_q^2}\right)\right];$$
$$a_2=\omega_q^2\omega_s^2;$$
$$b_0=a_{\psi u}\left\{1-(a_{\psi s}a_{s\zeta}+a_{qs}a_{sq}+a_{qs}a_{sq}a_{\psi q})+\frac{a_{qu}}{a_{\psi u}}\times\right.$$
$$\left.\times[a_{\psi q}(-1+a_{\zeta s}a_{s\zeta}+a_{\psi s}a_{s\psi})+a_{\psi s}a_{sq}]+\frac{a_{s\zeta}a_{\zeta u}}{a_{\psi u}}(a_{\psi s}-a_{\psi q}a_{qs})\right\};$$

$$b_1=\omega_q^2 a_{\psi u}\left\{2-a_{\zeta s}a_{s\zeta}+a_{\psi s}a_{s\zeta}\frac{a_{\zeta u}}{a_{\psi u}}-\left(1-\frac{a_{\psi u}a_{qu}}{a_{\psi u}}\right)\left(1-\frac{\omega_s^2}{\omega_q^2}\right)\right\};$$

$$b_2=a_{\psi u}\omega_q^2\omega_s^2.$$

Let us introduce the following dimensionless parameters:

$$\varkappa=1-\frac{\omega_s^2}{\omega_q^2};\quad \bar{a}=\frac{a}{(m^0+m)\,\eta^2(x_Q)};\quad \varepsilon=\frac{k}{\bar{a}};$$
$$l=\sqrt{\frac{J^0+J}{m^0+m}};\quad k=\frac{l\eta'(x_0)}{\eta(x_Q)}Z_0;\quad Z_0=\frac{l}{x_O-x_Q}. \tag{3.99}$$

Let us denote

$$a_0=\overset{*}{a}_0;\quad b_0=\overset{*}{b}_0;$$
$$a_1=\omega_q^2(\overset{*}{a}_1-\varkappa);\quad b_1=\omega_q^2\left[\overset{*}{b}_1-a_{\psi u}\varkappa(1-\varepsilon)\right]; \tag{3.100}$$
$$a_2=\omega_q^4(1-\varkappa);\quad b_2=\omega_q^4\overset{*}{b}_2(1-\varkappa),$$

where χ is a small parameter.

As the new characteristic frequency let us introduce the value

$$\omega_0^2=\frac{1}{2}(\omega_q^2+\omega_s^2)\cong\omega_q^2\left(1-\frac{\varkappa}{2}\right);\quad \omega_0^4\cong\omega_q^4(1-\varkappa). \tag{3.101}$$

Considering expression (3.101) we obtain

$$a_1=\omega_0^2\overset{*}{a}_1;\quad b_1=\omega_0^2\overset{*}{b}_1;$$
$$a_2=\omega_0^4\overset{*}{a}_2;\quad b_2=\omega_0^4\overset{*}{b}_2. \tag{3.102}$$

In the given case the criterion of structural stability of the space vehicle as applied to the auxiliary oscillators s and q also has the form

$$\Delta_4>0 \tag{3.103}$$

or

$$\Psi(Z_1,\ Z_2,\ k)=\left\{\left[(Z_1+1)^2-\frac{1}{\gamma}\right]Z_2-(Z_1+1)\right\}\left[\left(Z_1^2-\frac{k^2}{\gamma}\right)Z_2+kZ_1\right]>0, \tag{3.104}$$

where

$$Z_1=\frac{Z_0}{l}(R\bar{c}+x_{01}-x_O);$$
$$Z_2=\frac{Z_0^2\left[(\bar{c}R-h)\,\eta'(x_r)+\eta(x_r)\right]}{(Z_0^2+Z_1^2)\,\eta(x_Q)};$$
$$\gamma=\frac{\varrho\bar{a}R^3\bar{\lambda}^2}{\mu(m^0+m)};\quad l=\sqrt{\frac{J^0+J}{m^0+m}};\quad Z_0=\frac{l}{x_O-x_Q};\ \bar{c}=\frac{\lambda_{01}}{\mu};$$

h is the depth that the fuel fills the compartment; x_T is the coordinate characterizing the position of the surface of the liquid in the selected coordinate system.

If we neglect the inertial relations between the elastic hull and the oscillating liquid, then the inequality (3.103) becomes the following:

$$\Psi(Z_1, k) = -kZ_1(Z_1+1) > 0. \qquad (3.105)$$

When k<0 with forward (relative to the metacenter) location of the measuring device from the inequality (3.105) we have the criterion of structural stability in the case of one compartment:

$$Z_1(Z_1+1) < 0. \qquad (3.106)$$

The phase stabilization is achieved here for phase lead of the control system on frequencies ω_s and ω_q. For the most standard case k>0 (the first tone of the elastic vibrations of the system with forward location of the measuring device), from the expression (3.105) we have:

$$Z_1(Z_1+1) < 0. \qquad (3.107)$$

Here the phase stabilization of the system is achieved with phase delay of the control system of ω_s and ω_q.

Let us return to the more exact criterion (3.103), from which we find the functions characterizing the boundaries of the structural stability of the space vehicle in the plane Z_1, Z_2:

$$Z_2^{(1)} = \frac{Z_1+1}{(Z_1+1)^2-\gamma^{-1}}; \quad Z_2^{(2)} = -\frac{kZ_1}{Z_1^2-k^2\gamma^{-1}}. \qquad (3.108)$$

The curves (3.108) intersect the x-axis at the unique point $Z_1=-1$ or $Z_1=0$ respectively. Both curves have horizontal asymptotes $Z_2=0$; for $Z_1\to\infty$, $Z_2^{(1)}\to 0$, $aZ_2^{(2)} \gtrless 0$ for $k \gtrless 0$, and the vertical asymptotes

$$Z_1 = -1 \pm \gamma^{-1/2} \quad (\text{curve } Z_2^{(1)}(Z_1));$$
$$Z_1 = \pm k\gamma^{-1/2} \quad (\text{curve } Z_2^{(2)}(Z_1)).$$

Obviously there are three critical values of $|k|$ for which either the left or right asymptotes of the curves $Z_2^{(1)}$ and $Z_2^{(2)}$ coincide:

$$|k_1| = 1+\sqrt{\gamma}; \quad |k_2| = 1-\sqrt{\gamma}; \quad |k_3| = \sqrt{\gamma}. \qquad (3.109)$$

Correspondingly, there are six characteristic cases of mutual arrangement of the regions of structural stability and instability in the plane Z_1, Z_2 for various decreasing values of the parameter $|k|$ for fixed $\gamma<0.25$:

$$|k|>1+\sqrt{\gamma};\ |k|=1-\sqrt{\gamma};\ |k|<\sqrt{\gamma};$$
$$|k|=1+\sqrt{\gamma};\ |k|=\sqrt{\gamma};\ |k|\to\infty.$$

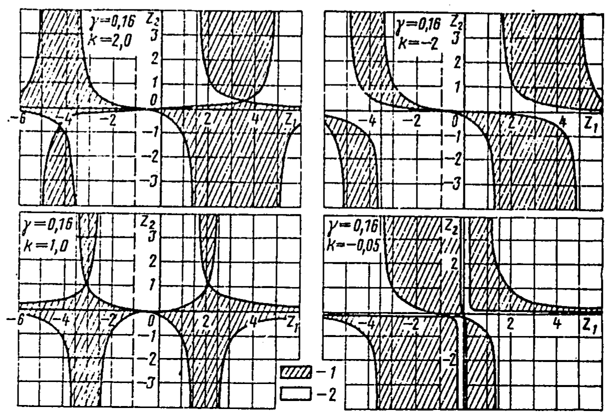


Figure 3.25. Regions of stabilizability (1), unstabilizability (2) considering the mutual effect of the oscillations of the fuel and the elasticity of the space vehicle hull

For $\gamma=0.25$ we have: $\sqrt{\gamma}=0{,}5;\ 1-\sqrt{\gamma}=\sqrt{\gamma}$.

Therefore when $\gamma=0.25$ the 3d and 4th cases change roles, and 5 corresponds to $|k|<1-\sqrt{\gamma}$.

In addition to the modulus of the parameter k, the form of the regions of stability essentially depends on the sign of the parameter so that each of the cases (3.110) corresponds to two versions $k>0$, $k<0$.

The standard cases are presented in Fig 3.25 ($\gamma=0.16$). The crosshatching corresponds to the region of structural instability ($\Psi<0$); the absence of crosshatching corresponds to the regions of structural stability ($\Psi>0$). From the performed analysis it follows that under defined conditions consideration of the elasticity of the structural elements of the space vehicle changes the conditions of stability of the closed system made up of the space vehicle and the automatic stabilization system on the oscillation frequency of the liquid: in cases where phase lead of the control system would be required for stability of the system, considering the elasticity phase delay is required, and vice versa.

The effect is more strongly manifested, the greater the connectedness of the corresponding partial systems.

Longitudinal Oscillations of the Space Vehicle Hull

The system of equations describing the longitudinal oscillations of the space vehicle will be considered as applied to the case where the frequencies of the natural longitudinal oscillations of the hull and the liquid in one of the lines are close. Omitting the indexes j and k corresponding to them, we obtain the following simplified system of equations:

$$\begin{gathered}\ddot{\xi}+a_{\xi s}\dddot{s}=a_{\xi u}u;\\ \ddot{q}+\beta_q\dot{q}+\omega_q^2q+a_{qs}\ddot{s}=a_{qu}u;\\ \ddot{s}+\beta_s\dot{s}+\omega_3^2s+a_{s\xi}\ddot{\xi}+a_{sq}\ddot{q}=0;\\ r+a_{rs}\omega_s^2s=0;\\ u=L(r).\end{gathered} \tag{3.110}$$

Considering the first of the equations (3.110) it is convenient to exclude the variable ξ, which leads to the system

$$\begin{gathered}\ddot{s}(1-a_{s\xi}a_{\xi s})+\omega_s^2s+a_{sq}\ddot{q}=-a_{s\xi}a_{\xi u}u;\\ \ddot{q}+\omega_q^2q+a_{qs}\ddot{s}=a_{qu}u;\\ r+a_{rs}\omega_s^2s=0;\\ u=L(r).\end{gathered} \tag{3.111}$$

Let us consider the problem of structural stability of the space vehicle as applied to two oscillatory elements characterized by the frequencies ω_s, ω_q analogously to how this was done earlier with respect to the oscillations of the fuel in two compartments with frequencies of σ_1, σ_2).

The characteristic equation of the closed system in the given case has the form

$$\Phi_0(p^2)+L(p)\Phi_k(p^2)=0, \tag{3.112}$$

where

$$\begin{aligned}&\Phi_0(p^2)=a_0p^4+a_1p^2+a_2;\quad \Phi_k(p^2)=b_1p^2+b_2;\\ &a_0=1-a_{s\xi}a_{\xi s}-a_{sq}a_{qs};\quad b_1=-a_{rs}(a_{s\xi}a_{\xi u}+a_{sq}a_{qu})\omega_s^2;\\ &a_1=(1-a_{s\xi}a_{\xi s})\omega_q^2+\omega_s^2;\quad b_2=-a_{rs}a_{s\xi}a_{\xi u}\omega_q^2\omega_s^2.\\ &a_2=\omega_q^2\omega_s^2.\end{aligned} \tag{3.113}$$

The criterion of structural stability of the system (3.111) has the form

$$Z_2[(-\varkappa+\gamma Z_1)k+Z_2(1-\varkappa-\gamma_k Z_1)]<0, \tag{3.114}$$

where

$$Z_1=\left(1+\frac{h}{l'}\right)^2; \quad Z_2=\frac{1+\frac{g^*}{\eta(x_p)}\,\frac{h}{l'}}{1+\frac{h}{l'}};$$

$$k=\frac{a}{(m^0+m)\,\eta^2(x_p)}; \quad \varkappa=1-\frac{\omega_s^2}{\omega_q^2}; \quad \gamma=\frac{m'}{m^0+m};$$

$\eta(x)$ is the considered form of the longitudinal oscillation; g^* is the absolute displacement of the apparent mass of the liquid in the tank, corresponding to the form $\eta(x)$; m' is the mass of the fuel in the lines; h is the depth of the fuel in the compartment; ℓ' is the length of the lines; a is the generalized mass corresponding to the investigated form of oscillations of the space vehicle hull; m^0+m is the total mass of the hull of the space vehicle and the fuel in the tanks.

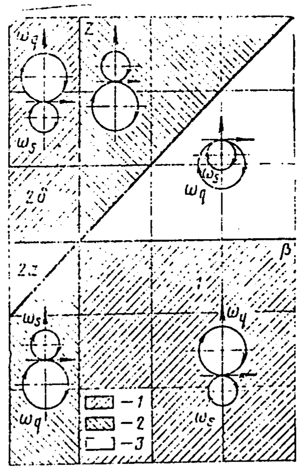


Figure 3.26. Regions of stabilizability (1), unstabilizability on the oscillation frequency of the hull (2), unstabilizability on the oscillation frequency of the fuel in the line (3) of the space vehicle

The relation of the indicated parameters of the dynamic system of the coefficients is presnted, for example, in reference [56].

Considering that by definition $Z_1 \ll \gamma^{-1}$, where $\gamma \ll 1$, let us simplify the inequality, by introducing the parameters

$$\beta=\frac{\omega_q^2}{\omega_s^2}-1=\frac{\varkappa}{1-\varkappa}\quad(-1\leqslant\beta\leqslant\infty);$$

$$Z=\frac{Z_2}{k}=\frac{\left[1+\frac{g^*}{\eta(x_p)}\right](m^0+m)\,\eta^2(x_p)}{\left(1+\frac{h}{l'}\right)a}. \qquad (3.115)$$

Thus, the criterion of structural stability of the investigated system takes on the following form:

$$\frac{k^2Z}{1+\beta}(Z-\beta)<0. \qquad (3.116)$$

The regions of structural instability in the plane Z,β are bounded by the straight lines Z=0; Z=β and they are presented in Fig 3.26.

The opposite direction of the crosshatching in Fig 3.26 corresponds to different signs of the first or second cofactors in the formula (3.116); the forward crosshatching corresponds to the instability on a frequency of ω_q; the return crosshatching, on the frequency ω_s; the absence of crosshatching corresponds to the structural stability (stability with a phase delay of the engine).

Increasing the parameter β leads to successive disturbance of the conditions of phase stabilization: for Z<0 first on a frequency close to ω_s and then on a frequency close to ω_q; for Z>0, on the contrary, which is illustrated by the phase-amplitude characteristics of the open system in the vicinity of the characteristic frequencies in Fig 3.26. The line β=0 is a special line.

In this chapter a study is made from the point of view of stabilizability of the mathematical models of space vehicles for certain standard conditions of motion in the active segment. The equations of motion which are basic for the analysis were borrowed from the monograph by B. I. Rabinovich [56].

The study of the stabilizability of the spacecraft in all cases is the study of the effect of control in the stabilization channels (yawing, pitch, heel) on the stability of the additional oscillatory elements (the fuel oscillations in the compartments, the elastic vibrations of the structural elements, and so on).

The analysis shows that obviously there is no compositional layout of the base vehicle or standard conditions of its movement in the active section for which the problem of stabilizing the object will reduce to phase stabilization of the space vehicle by insuring lead (or delay) of the phase characteristic of the control system in the investigated frequency range.

As is obvious, frequently more exact tuning of the control system is required (in the case of structural instability of the object), sometimes it is impossible in general to stabilize the object by adjusting only the "phase" of the control system (in the case of dynamic instability of the space vehicle).

Accordingly, the following two areas of investigation arise:

The study of the possibility of optimizing the compositional layout of the space vehicle in the design phase in order to crease an object which is improved in the dynamic sense not imposing increased requirements on the structure and characteristics of the stabilization system.

The investigation of the possibility of amplitude stabilization of the structurally unstable space vehicles with the use of such effective procedures as the introduction of dampers, simultaneous tuning of both the phase and the dynamic amplification coefficient of the control system on defined frequencies and also efficient solution of the stabilization algorithm of the object. These two areas determined the group of problems investigated in the following chapter.

CHAPTER 4. APPLICATION OF THE THEORY OF STABILIZABILITY TO THE PROBLEMS OF SPACE VEHICLE DESIGN

4.1. Investigation of the Structural Properties of the Designed Space Vehicles

General Remarks

This section contains a discussion of the algorithm for analyzing the structural properties of designed space vehicles considering the variability of mass caused by burning off fuel. The study is made as applied to the oscillation mode of the space vehicle relative to the center of masses with the angular stabilization system included (US).

A study is made of the classical case of a two-tank compositional space vehicle system providing for the placement of a two-component fuel (oxidizing agent and combustible fuel component) required for operation of the rocket engine. Fig 4.1 shows an example of a standard compositional layout for such a space vehicle. In the given case the tank with the oxidizing agent has a spherical shape, the tank with the combustible fuel component, toroidal. Their placement with respect to the longitudinal axis is characterized by the coordinates x_{01}, x_{02}. In addition, in this figure the coordinates of the metacenter (x_G) and the coordinates of the location of the controlling engines (x_p) are denoted.

Hereafter the assumption will be made that the fuel tanks have arbitrary (axisymmetric) configuration.

As was demonstrated in the preceding chapter, each combination of characteristic parameters of the space vehicle β, γ, ζ, c, k corresponds entirely to the defined configurations of the boundaries of the regions of stabilizability in the Z_1, Z_2 plane. It is characteristic that for all the variety of compositional systems of the space vehicle the number of types of curves limiting these regions is finite (straight lines, ellipses, defective hyperbolas).

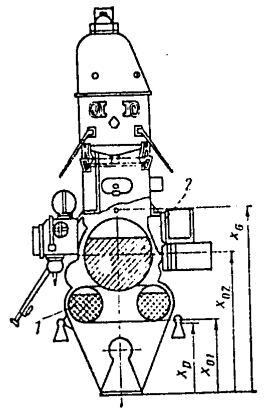


Figure 4.1. Standard diagram of the placement of the fuel tanks in the hull of a spacecraft:
1 -- combustible fuel component; 2 -- oxidizing agent

Let us introduce the parameter $\tau=t/T$ into the investigation, where t is the dynamic time of the active segment of the space vehicle and T is the total time of the active segment of the space vehicle ($0\leq\tau\leq1$). Then the boundaries of the regions of stabilizability in the Z_1, Z_2 plane can be considered as projections of the corresponding boundary surfaces in the space τ, Z_1, Z_2 on the plane $\tau=\tau^*$, where τ^* is the appropriately selected time τ (Fig 4.2, a, b.

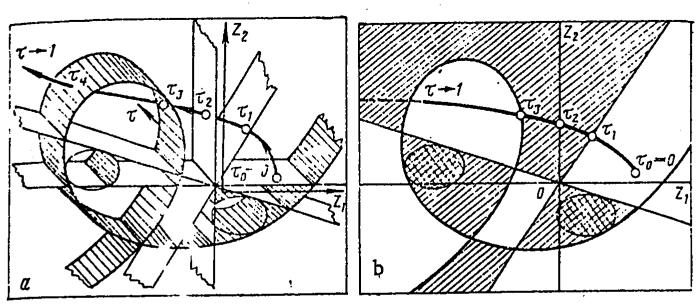


Figure 4.2. Regions of stabilizability of the space vehicle (see Fig 4.1) with toroidal and spherical fuel compartments:
a -- the space of the parameters Z_1, Z_2, τ; b -- in the plane of the parameters Z_1, Z_2

Let us fix the values of $Z_1(0)$, $Z_2(0)$. In the space Z_1, Z_2, τ let us obtain a line Γ of the states on the space vehicle determined by the parametric equations

$$Z_1 = Z_1(\tau); \quad Z_2 = Z_2(\tau),$$

the position of which with respect to the regions of stabilizability provides information about the dynamic properties of the space vehicle in the active segment.

In many cases, the situation is such that on variation of the parameter τ the boundaries of the regions of stabilizability in the Z_1, Z_2 plane are in practice stationary, and the variation of the position of the characteristic point $\Gamma[Z_1(\tau^*), Z_2(\tau^*)]$ of the space vehicle with respect to the regions is connected only with its movement along the curve $\Gamma(\tau)$ for $0 \leq \tau \leq 1$. In this case the problems of the construction of the region and the characteristic curve $\Gamma(\tau)$ are separated, and the possibility is presented of estimating the quality of the various compositional systems in general form.

Thus, on the whole the problem of analyzing the dynamic properties of a space vehicle reduces to a geometric problem -- the investigation of the configuration of the regions of stabilizability and the placement of the characteristic lines $\Gamma(\tau)$ in space (Z_1, Z_2, τ) or in the plane (Z_1, Z_2). Here it is expedient to adhere to the following sequence of operations:

Calculation of the dimensionless parameters of the object for the characteristic points $\tau=0$; $\tau=\tau_j, \ldots, \tau=\tau_n=1$;

Calculation of the boundaries of the regions of natural and structural instability of the object

$$D(Z_1, Z_2, \tau) = 0, \quad \Psi(Z_1, Z_2, \tau) = 0$$

in the space of the parameters Z_1, Z_2, τ; here it is necessary to isolate the segments (τ_s, τ_{s+1}), the boundaries of which correspond to vanishing of one of the parameters β, c (for $\beta=0$, $c=0$ the topological structure of regions of structural instability varies);

Calculation of the time function $D(\tau)$, $\Psi(\tau)$ and investigation of the sign of these functions for $0 \leq \tau \leq 1$;

Structure of the projections of the lines $\Gamma(\tau)=\{Z_1(\tau), Z_2(\tau)\}$ and the regions of stability on the plane τ=const.

Analysis of the Stabilizability of a Space Vehicle in Various Design Phases

a) General Data on the Object are Known

Let the following be given:

Geometric characteristics: shape of the fuel tanks and their arrangement, elongation of the object, location of the servoelements, and so on.

Physical characteristics: density of the fuel components, mass distribution, alignment, moment of inertia with respect to the center of mass, and so on.

Let us also propose that the dimensionless hydrodynamic coefficients are known which depend on the form of the tanks and depth of filling them with the fuel:

$$\bar{\Omega}_i,\ \bar{c}_i=\frac{\lambda_{0i}}{\lambda_i};\ \bar{M}_i=\frac{\lambda_i^2}{\mu_i},\ \bar{\sigma}_i^2(i=1,\ 2).$$

For the majority of configurations which are widespread at the present time these coefficients have been calculated and published. In other, more complex cases, they can be calculated by approximate formulas.

The characteristic parameters of the object Z_1, Z_2, k, β, γ, ζ, c are related to the geometric and physical parameters of the object by the following expressions:

$$Z_i=\frac{1}{l}\left[R_{0i}\bar{c}_i+x_G-x_{0i}\right];\quad l=\sqrt{\frac{J^0+J}{m^0+m}};$$

$$c=\frac{l}{x_G-x_p};\ k=\frac{\sigma_2^2\bar{M}_1\varrho_{01}R_{01}^3}{\sigma_1^2\bar{M}_2\varrho_{02}R_{02}^3};\ \zeta=\frac{R_{01}}{l\sigma_1^2};$$

$$\gamma=\bar{M}\,\frac{\sigma_1^2\varrho_{02}R_{02}^3}{\sigma_2^2\varrho_{02}R_{01}^3};\ \beta=\frac{1}{k\gamma}\left(-1+\frac{\sigma_2^2}{\sigma_1^2}\right),$$

where on the basis of the notation that has been adopted R_{0i}, x_{0i} are the characteristic size and coordinate of the characteristic point of each of the tanks; ρ_{0i} is the density of the fuel components in the i-th tank; x_p is the coordinate of the point of application of the control input; x_G is the coordinate of the metacenter (all the coordinates are reckoned from the plane of the tip of the nozzle of the sustainer engine where the origin of the coordinate systems bound to the object is placed).

As the first example, let us consider a spacecraft, the two fuel tanks of which have a form close to cylindrical.

Let us set:

$$\bar{x}_{01}=\frac{x_{01}}{R_{01}}=1{,}45;\ \ \bar{x}_{02}=\frac{x_{02}}{R_{02}}=3{,}2;\ \ \bar{x}_p=\frac{x_p}{R_{01}}=0{,}37;\ \ \frac{\varrho_{02}}{\varrho_{01}}=2{,}0.$$

The other initial data, which are a function of the depth to which the tanks are filled with fuel depending in turn on the time τ=t/T (where T is the total time of the active segment) are presented in Table 4.1. The results of calculating the natural and structural stability of the space vehicle are presented in Table 4.2. As follows from these data, the dimensionless parameters $c(\tau)$, $k(\tau)$, $\gamma(\tau)$, $\zeta(\tau)$, $\beta(\tau)$ and, consequently, the boundaries of regions of dynamic instability

$$D[Z_1,\ Z_2;\ k(\tau),\ \gamma(\tau),\ \zeta(t)\beta(\tau)]=0;$$
$$\Psi[Z_1,\ Z_2 c(\tau),\ k(\tau),\ \gamma(\tau),\ \zeta(\tau),\ \beta(\tau)]=0$$

are slightly deformed in the investigated time interval $0.1 \lesssim \tau \lesssim 1$. Therefore in accordance with the proposed approach let us consider the projection of the regions of stability and the characteristic line $\Gamma(\tau)$ in the space Z_1, Z_2, τ on the plane $\tau=0.3$. As is obvious from Fig 4.3, a, the plane Z_1, Z_2 is broken down into three types of regions by the set of curves $D(Z_1, Z_2)=0$; $\Psi(Z_1, Z_2)=0$:

The regions of stabilizability of the object $\Psi(\tau)>0$; $D(\tau)>0$. The stabilization of the object in the active section is possible by rough adjustment of the parameters of the automatic stabilization system:

$$\operatorname{sign}[L(i\omega_j)]=\text{const},\ j=1,\ 2;$$

Table 1

τ	$\bar{h}_1$	$\bar{M}_1$	$\bar{c}_1$	$\bar{\Omega}_1$	$\bar{h}_2$	M_2	$\bar{c}_2$	$\bar{\Omega}_2$	$\bar{\sigma}_1^2$	σ_2^2	l
0,1	1,29	1,43	1,28	3,96	1,60	1,43	1,88	3,96	1,83	1,84	2,40
0,3	0,98	1,41	1,07	3,96	1,33	1,43	1,51	3,96	1,82	1,84	2,63
0,45	0,80	1,39	0,88	3,96	1,06	1,42	1,16	3,96	1,80	1,83	2,86
0,75	0,62	1,34	0,75	3,96	0,80	1,39	0,88	3,96	1,72	1,80	3,15
0,9	0,45	1,20	0,72	3,96	0,53	1,18	0,72	3,96	1,55	1,66	3,57
1,0	0,25	0,89	0,88	3,96	0,30	0,89	0,88	3,96	1,15	1,15	4,24

Table 2

τ	Z_1	Z_2	c	K	ζ	β	γ	γ_1	γ_2	sign Ψ	sign Φ
0,1	0,80	−0,62	0,44	2,02	0,33	0,018	0,09	−0,49	1,01	−	+
0,3	0,79	−0,45	0,49	2,05	0,32	0,036	1,10	−0,48	1,02	−	+
0,45	0,81	−0,24	0,54	2,07	0,29	0,067	0,12	−0,47	1,04	+	+
0,75	0,86	−0,025	0,57	2,10	0,28	1,130	0,16	−0,45	1,09	+	+
0,9	0,87	0,15	0,61	2,16	0,27	0,160	0,20	−0,45	1,11	+	+
1,0	0,96	0,35	0,60	2.02	0,31	0	−0,25	−0,45	1,00	+	+

The region of structural instability of the object $\Psi(\tau)<0$; $D(\tau)>0$ (crosshatched). In order to insure stability of a closed system "fine" tuning of the parameters of the automatic stabilization system is required on the basis of the contradiction of the requirements on $\phi(\omega)$ on frequencies of $\omega=\sigma_1$; $\omega=\sigma_2$;

The regions of dynamic instability (double crosshatched): $\Psi(\tau)<0$; $D(\tau)<0$. The stability of the closed system cannot be insured by selection of the parameters of the automatic stabilization system of the adopted structure.

This case must be considered unfavorable because the closed system made up of the space vehicle and the automatic stabilization system is unstable.

In order to eliminate instability one of the fuel tanks must be equipped with special damping devices insuring the required decrements of the fuel oscillations in the tanks at the beginning of the active segment $0<\tau\leq 0.3$ which depend on the specific parameters of the automatic stabilization system.

In the initial steps of the projection it is necessary to also consider the possibility of rearrangement of the object, eliminating the structural instability.

b) The coefficients of the equations of the disturbed movement of the space vehicle are known.

The characteristic parameters Z_1, Z_2, k, β, γ, ζ, c have been calculated in the given case by the formulas

$$Z_1=\frac{1}{l}\,\frac{a_{s\psi}}{a_{s_1 z}};\ Z_2=\frac{1}{l}\,\frac{a_{s_2\psi}}{a_{s_2 z}};\ l=\sqrt{\frac{a_{s_1\psi}a_{zs_1}}{a_{s_1 z}a_{\psi s_1}}};$$

$$\gamma=a_{zs_1}a_{s_1 z};\ \zeta=-\frac{a_{z\psi}}{l\sigma_1^2};$$

$$\beta=\frac{-1+\frac{\sigma_2^2}{\sigma_1^2}}{a_{zs_2}a_{s_2 z}};\ k=\frac{a_{zs_2}a_{s_2 z}}{a_{zs_1}a_{s_1 z}};\ c=\frac{1}{l}\,\frac{a_{zu}}{a_{\psi u}}.$$

The criteria of dynamic stability of the space vehicle and its structural stability have the following form, respectively

$$D=a_1^2-4a_0a_2>0;$$

$$\Psi=(a_1b_2-b_1a_2)(a_0b_1-a_1b_0)-(a_0b_2-b_0a_2)^2>0,$$

where

$$a_0=1-\mu_1+\mu_2;\quad b_0=1-\mu_{3u};$$

$$a_1=2-\mu_1+\bar{\mu}_2+\mu_3;\quad b_1=2-\mu_{3u}+\mu_5;$$

$$a_2=1+\bar{\mu}_2+\mu_4;\qquad b_2=\mu_6.$$

The parameters μ_i are related to the coefficients of the system of equations of disturbed motion (3.20) by the following expressions:

$$\mu_1=a_{s_1 z}a_{zs_1}+a_{s_2 z_1}a_{zs_1}+a_{\psi s_1}a_{s_1\psi}+a_{\psi s_2}a_{s_2\psi};$$

$$\mu_2=(a_{s_1\psi}a_{s_2 z}-a_{s_2\psi}a_{s_1 z})\,(a_{\psi s_1}a_{zs_2}-a_{zs_1}a_{\psi s_2});$$

$$\bar{\mu}_2 = -\frac{a_{z\psi}}{\sigma_1^2}\left[a_{\psi s_1} a_{s_1 z} + a_{\psi s_2} a_{s_2 z}\right];$$

$$\mu_3 = \left(-1 + \frac{\sigma_2^2}{\sigma_1^2}\right)\left(1 - a_{s_1\psi} a_{\psi s_1} - a_{s_1 z} a_{z s_1}\right);$$

$$\mu_4 = \left(-1 + \frac{\sigma_2^2}{\sigma_1^2}\right)\left(1 - \frac{a_{z\psi}}{\sigma_1^2} a_{s_1 z} a_{\psi s_1}\right);$$

$$\mu_{3u} = a_{s_1 z} a_{z s_1} + a_{s_2 z} a_{z s_2} - \frac{a_{zu}}{a_{\psi u}}\left(a_{s_1 z} a_{\psi s_1} + a_{s_2 z} a_{\psi s_2}\right);$$

$$\mu_5 = \left(-1 + \frac{\sigma_2^2}{\sigma_1^2}\right)\left(1 - a_{s_1 z} a_{z s_2} + \frac{a_{zu}}{a_{\psi u}} a_{s_1 z} a_{\psi s_1}\right);$$

$$\mu_6 = \frac{\sigma_2^2}{\sigma_1^2}.$$

As the next example let us consider the spacecraft for which the coefficients of the equations of undisturbed motion are presented in Table 4.3.

Table 4.3

τ	$a_{s_2\psi}$	$a_{s_2 z}$	$a_{s_1\psi}$	$a_{s_1 z}$	$a_{\psi s_2}$	$a_{z s 2}$	$a_{\psi s_1}$
0,05	3,35	−1,54	3,88	−1,54	0,01	−0,023	0,002
0,50	−5,49	−1,54	−4,56	−1,54	−0,016	−0,033	−0,003
0,90	−16,74	−1,50	−16,53	−1,54	−0,072	−0,052	−0,013

τ	$a_{z s_1}$	a_{zu}	$a_{\psi u}$	$a_{z\psi}$	σ_2^2	σ_1^2
0,05	−0,004	−5,18	−1,00	−10,37	9,32	19,12
0,50	−0,006	−7,30	−1,06	−14,61	13,14	26,95
0,90	−0,010	−11,47	−1,88	−22,94	20,17	42,38

The results of calculating the stabilizability of the space vehicle are presented in Table 4.4 and Fig 4.3, b.

Table 4.4

τ	Z_1	Z_2	c	k	ζ	β	γ	η_1	η_2	sign Ψ	sign D
0,05	−0,35	−0,30	0,72	0,09	0,07	−14,02	0,07	0,01	−13,12	+	+
0,50	0,35	0,42	0,82	0,09	0,06	−9,96	0,10	0,02	−9,06	+	+
0,90	1,19	1,23	0,67	0.09	0,06	−6,65	0,16	0,02	−5,76	+	+

As is obvious, in the given case the boundaries of the regions of stable instability of the object are close to straight lines (which is explained by the smallness of the parameter β).

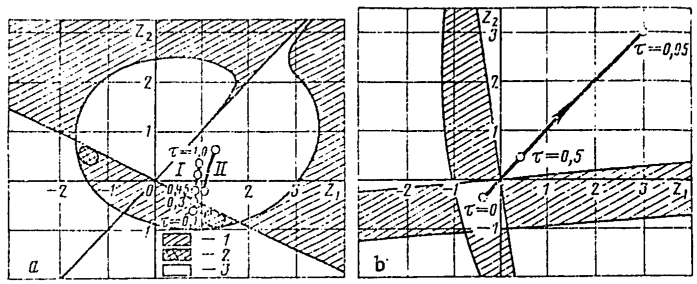


Figure 4.3. Regions of stabilizability (1), unstabilizability (2) of the space vehicle with cylindrical fuel compartments:
a -- for example 1; b -- for example 2

The object is stabilized in the entire active segment, which in the presence of lead of the automatic stabilization system in the frequency range of σ_i, i=1,2 indicates stability of the closed system in the generally accepted sense.

4.2. Stabilizability Criterion as the Quality Criterion of the Compositional Layout of a Space Vehicle

Study of the Effect of the Compositional Layout of a Space Vehicle on Its Structural Properties

Let us return to the investigation of the possibility of the projection of structurally stable objects by selecting the corresponding compositional system. The basic problem which arises here is the problem of the effect of the compositional system, the form and the location of the fuel tanks, the alignment, the form and location of the control elements, and so on on the structural properties of the spacecraft.

This problem is partially investigated in Chapter 3 where, however, primary attention was given to the configuration of the regions of stability of the space vehicle as a function of the values of the characteristic parameters of the object.

From the practical point of view it is necessary to study also the mutual arrangement of the characteristic line $\Gamma(\tau)$ and the corresponding regions for the standard configurations of the fuel tanks.

Each compositional layout of the space vehicle is characterized from the ponit of view of the discussed method:

By the regions of instability of one of several possible configurations; the location of the regions in the Z_1, Z_2 plane, which depends on the specific characteristics of the object;

A characteristic line $\Gamma(\tau)$ given by the parametric equations

$$Z_1 = Z_1(\tau); \quad Z_2 = Z_2(\tau),$$

where τ is the current time of the active segment.

The problem of analyzing the dynamic properties of the spacecraft reduces to a geometric problem -- the investigation of the mutual arrangement of the line $\Gamma(\tau)$ and the regions of dynamic stability corresponding to the given composition.

As an example, let us consider a spacecraft with liquid-propellant rocket engine having spherical, cylindrical, conical or toroidal fuel tanks in the following combinations: sphere-sphere; sphere inside sphere, cylinder-cylinder; torus-torus; torus-sphere; torus-cone.

The example fuel tanks correspond to the required mass ratio for the fuel vapor including hydrogen tetroxide N_2O_4 as the ozidizing agent (density $\rho_0=1.45 \cdot 10^{-3}$ kg/m^3), and aerosin-50 as the combustible fuel component (the density $\rho=0.9 \cdot 10^{-3}$ kg/m^3).

Assuming that the mass of the fuel at the time the sustainer is switched on is half the mass of the spacecraft, it is easy to calculate the characteristic parameters k, γ, β, ζ in the form of time functions of the active segment τ. Let us assume that the control of the space vehicle is realized by a pair of forces collinear with the longitudinal axis of the object, and let us consider the asymptotic behavior of the curve defined by the equations

$$Z_l = -\frac{1}{l(\tau)} \left[R_{0i} \bar{c}_i(\tau) + x_{0l} - x_G(\tau) \right]$$

for $\tau \to 0$ and $\tau \to 1$.

Let us consider the parameters corresponding $\tau=0$ and $\tau=1$ and let us assign the indexes 0 and 1 respectively. Setting $h_i(\tau) \to 0$ for $\tau \to 1$, we have

$$l_0 = \sqrt{\frac{J^0 + J}{m^0 + m}}; \quad x_{G_0} = x_{\tau_0} + \sum_{l=1}^{2} \frac{\bar{\Omega}_{l0} \Omega_{0l} R_{0l}}{m^0 + m};$$

$$l_1 = \sqrt{\frac{J^0}{m^0}}; \quad x_{G_1} = x_{\tau 0};$$

$$-\bar{h}_{0l} + \frac{1}{\bar{\omega}_l^2} < \bar{C}_{l0} < \bar{h}_{0l} + \frac{1}{\bar{\omega}_l^2}.$$

Let us proceed directly to the selected compositional diagrams.

1. The spherical tanks (the index "1" corresponds to the tank with the oxidizing agent).

$$z_{i0}=\frac{1}{l_0}(x_{G_0}-x_{0i});\quad Z_{i1}=\frac{1}{l_1}(x_{G_1}-x_{0i}).$$

In the given case the parameters Z_i are positive for the "lower" position of the tanks with respect to the metacenter and negative for the "upper" position. The relative position of the line $\Gamma(\tau)$ and the boundaries of the regions of stability in the case of a body with intermediate position of the metacenter is shown in Fig 4.4, a.

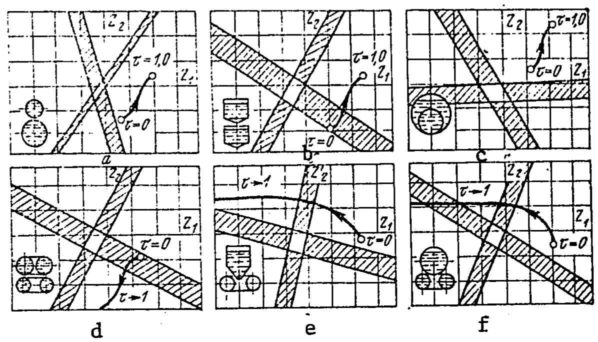


Figure 4.4. Mutual arrangement of the characteristic line under the regions of stabilizability for the space vehicles of different compositional systems

2. The spherical tank with the oxidizing agent in which the smaller size spherical tank with the combustible fuel component is placed (the index "1" corresponds to the tank with the combustible fuel component, Fig 4.4., b):

$$Z_{10}=\frac{1}{l_0}(x_{G_0}+x_{01});\quad Z_{11}=\frac{1}{l_1}(x_{G_1}-x_{01});$$
$$Z_{20}=\frac{1}{l_0}(x_{G_0}-x_{02});\quad Z_{21}=\frac{1}{l_1}(x_{G_1}-x_{02}-R_{01}\bar{C}_{10}).$$

3. Cylindrical tank with spherical bottoms (in cases 3-6 the index "1" corresponds to the tank with the combustible fuel component, Fig 4.4, c):

$$Z_{i0}=\frac{1}{l_0}\left(x_{G_0}-x_{0i}-R_{0i}\bar{h}_i+\frac{R_{0i}}{\xi}\right);\quad Z_{i1}=\frac{1}{l_1}(x_{G_1}-x_{01}).\quad (i=1,2)$$

4. Toroidal tanks (Fig 4.4, d):

$$Z_{i0}=\frac{1}{l_0}(x_{G_0}-x_{0i}-R_{0i}\bar{C}_{i0});\quad Z_{i1}=\frac{1}{l_1}(x_{G_1}-x_{0i}-R_{0i}\bar{C}_{i0}).$$

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5. Toroidal and spherical tanks (Fig 4.4, e):

$$Z_{10}=\frac{1}{l_0}(x_{G_0}-x_{01}-R_{01}\bar{C}_{10});\quad Z_{11}=-\frac{R_{01}}{l\bar{\omega}_1^2};$$

$$Z_{20}=\frac{1}{l_0}(x_{G_0}-x_{02});\quad Z_{21}=\frac{1}{l_1}(x_{G_1}-x_{02}).$$

6. Toroidal and conical tanks (Fig 4.4, f):

$$Z_{10}=\frac{1}{l_0}(x_{G_0}-x_{01}-R_{01}\bar{C}_{10});\quad Z_{11}=\frac{1}{l_1}(x_{G_1}-x_{01}-R_{01}\bar{C}_{10});$$

$$Z_{20}=\frac{1}{l_0}(x_{G_0}-x_{02}-R_{02}\bar{C}_{20});\quad Z_{21}=\frac{1}{l_1}(x_{G_1}-x_{02}).$$

The analysis of the mutual arrangement of the lines $\Gamma(\tau)$ and the regions of instability for the investigated compositional systems of Fig 4.4 indicates that the dynamic properties of the space vehicle essentially depend on the configuration and mutual arrangement of the fuel tanks. In particular, the space vehicles having at least one toroidal tank (in general, the bi-connected cavity) cannot be made structurally stable in the entire active segment. In practice this means that the use of the toroidal tanks predetermines the necessity for structural development of the space vehicle, namely, the installation of spatial damping devices to insure stability of the equipment in the nonstabilizability segments.

The space vehicles having spherical, cylindrical and other analogous tanks of the singly connected cavity type can be both stabilizable and unstabilizable in the active segment. The result depends to a great extent on the alignment of the object. Therefore this problem requires further investigation.

Optimization of the Structural Parameters of the Compositional System of the Spacecraft in the Early Stages of Design

Let us first consider the problem of the effect of the alignment of the space vehicle on its dynamic stability and the corresponding possibilities for optimization.

It is known that the position of the center of masses of the flight vehicle determines its static characteristics. From the analysis performed above it follows that the dynamic characteristics of the space vehicle also essentially depend on the alignment.

Let us consider this problem from the point of view of the possibility of improvement of the dynamic characteristics of the space vehicle, taking x_T as the variable parameter. As the object of investigation let us take the space vehicle with spherical fuel tanks.

In the given case we have

$$Z_1(\tau)=\frac{1}{l(\tau)}[x_G(\tau)-x_{01}],\quad Z_2(\tau)=\frac{1}{l(\tau)}[x_G(\tau)-x_{02}].$$

Excluding the coordinate of the metacenter from the two equations, we obtain

$$Z_2=Z_1-\frac{x_{02}-x_{01}}{l(\tau)}.$$

In the case of $\ell(\tau)$=const, the characteristic line is a straight line parallel to the bissectrix of the coordinate angle $Z_1 0 Z_2$. In the case of an increasing function $\ell(\tau)$ the line $\Gamma(\tau)$ deviates from the straight lines counterclockwise; for a decreasing function, clockwise.

The coordinates of the initial points of the curve $\Gamma(\tau)$, as is obvious from formulas (4.2), (4.3), essentially depend on the position of the metacenter of the space vehicle, related to the coordinate of the center of masses of the space vehicle by the equation

$$x_G=x_\tau+\sum_{i=1}^{2}\frac{\varrho_i\bar{\Omega}_i R_{0i}}{m^0+m}.$$

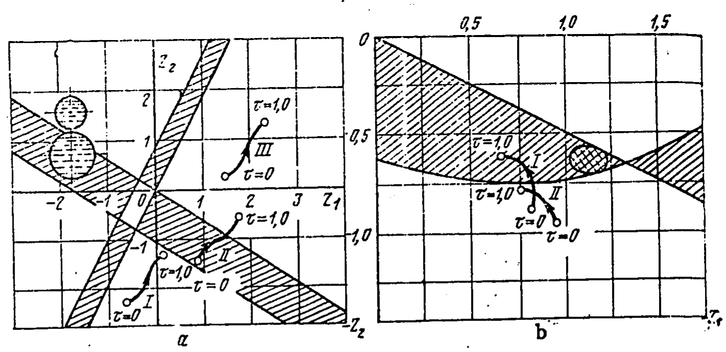

Figure 4.5. Optimization of the dynamic properties of the space vehicle by varying the alignment (a) and mutual position of the fuel compartment (b)

The following cases are possible:

The "rear" alignment $x_G(\tau)<x_{01}<x_{02}$ (the characteristic lines depicted in Fig 4.5, a, curve I);

The "intermediate" alignment $x_{01}<x_G(\tau)<x_{02}$ (curve II in Fig 4.5, a);

"Forward" alignment $x_{01}<x_{02}<x_G(\tau)$ (curve III in Fig 4.5, a).

As is obvious from Fig 4.5, a, the position of the metacenter and, consequently the center of masses can be used as the controlling parameter for variation of the dynamic properties of the space vehicle within the framework of the selected compositional system. In the given case in order to eliminate the structural instability, it is necessary to shift the center of masses of the space vehicle to the extreme upper (lower) position.

Analogous arguments can be made also for other compositional systems, in particular, in the case of cylindrical tanks. Let us again return to the first example and section 4.1 in order to obtain specific recommendations with respect to the variations of alignment of the object eliminating the unstabilizability of the object (see Fig 4.3, a).

In the case of cylindrical tanks and sufficient depth of the liquid $h_i \geq 2R_{0i}$, which is satisfied for the investigated time interval of unstabilizability of the object ($0 \lesssim \tau \lesssim 0.3$), the dimensionless parameters, according to the expression given above, have the form

$$Z_i = \frac{1}{l}\left[x_0 - x_{0i} + 0{,}54R_{0i}\right],\ i = 1,\ 2;$$

$$\beta = 0;\ k = \frac{\varrho_{02}}{\varrho_{01}};\ \zeta = 0{,}54\frac{R_{01}}{l};\ c = \frac{l}{x_0 - x_p}.$$

The angular coefficient of the straight line $Z_2 = \eta_1 Z_1$ is equal to k and does not depend, therefore, on time.

In order to derive the curve $\Gamma(\tau)$ from the region of structural instability crosshatched in Fig 4.3,a, obviously it is necessary to decrease the value of Z_{20}. It is easy to see that the required effect will be achieved if the center of masses of the system is shifted forward by the corresponding amount, keeping its radius of inertia invariant (curve II).

Thus, the performed analysis indicates that there is a real possibility for improving the dynamic characteristics of the closed system made up of the space vehicle and the automatic stabilization system by selection of an efficient compositional system and alignment in the drawing stage.

The criteria obtained for natural and structural stability can be used as the quality criteria of the compositional system if the corresponding functional is appropriately introduced.

Let us discuss one of the possible formal systems.

Let us introduce into the investigation the functions of the dimensionless parameters ν_k (k=1,2,...,r) of the object of control $D(\nu_1,\ldots,\nu_\ell)$, $\Psi(\nu_1,\ldots,\nu_\ell)$, having the property that the equality $\Psi(\nu_1,\ldots,\nu_\ell)=0$ corresponds to the boundary of the structural stability, and the equality $D(\nu_1,\ldots,\nu_\ell)=0$, the boundary of the natural dynamic stability of the object of control.

The parameters ν_k of the spacecraft are determined primarily by the geometric configuration of the fuel tanks of the spacecraft and their position and also the ratio of the mass densities of the fuel component. These parameters are different; the most varied geometric configurations are possible: cylindrical, spherical, toroidal, and so on in all possible combinations.

Let us assume that the parameters ν_k, in turn, are functions of the parameters p_i $(i=1,2,\ldots,n)$, q_j $(j=1,2,\ldots,m)$ and satisfy the equation

$$\begin{aligned} &\alpha_i \leqslant p_i \leqslant \beta_i && (i=1,\ 2,\ldots,\ n); \\ &q_j^0 \leqslant q_j \leqslant q_j^0 + \Delta q_j^0 && (j=1,\ 2,\ldots,\ m), \end{aligned} \tag{4.1}$$

where $\Delta q_j{}^0$ are small positive values, the squares and higher powers of which can be neglected. Thus,

$$\nu_k = \nu_k(p_1,\ p_2,\ldots,\ p_n,\ q_1,\ldots,\ q_m),\ k=1,\ 2,\ldots,\ r.$$

Then let us assume that there are values of $\nu_\ell{}^0$ for which the following inequalities are satisfied:

$$\begin{aligned} &D(\nu_1^0,\ \nu_2^0,\ldots,\ \nu_l^0) < 0; \\ &\Psi(\nu_1^0,\ \nu_2^0,\ldots,\ \nu_l^0) < 0. \end{aligned} \tag{4.2}$$

Let us introduce the linear coupling of the functional of the following type into the investigation

$$I = \lambda_1 \int_0^1 D(\nu_1,\ldots,\ \nu_r)\delta_1(\tau)\,d\tau + \lambda_2 \int_0^1 \Psi(\nu_1,\ldots,\ \nu_2)\delta_2(\tau)\,d\tau, \tag{4.3}$$

where λ_1 and λ_2 $(0\leq\lambda_i\leq1,\ i=1,2)$ are fixed weight coefficients not vanishing simultaneously, the ratio of which depends on the specific problem:

$$\delta_1(\tau) = \begin{cases} 0 \text{ for } D > 0; \\ 1 \text{ for } D \leqslant 0; \end{cases} \quad \delta_2(\tau) = \begin{cases} 0 \text{ for } \Psi > 0, \\ 1 \text{ for } \Psi \leq 0. \end{cases}$$

The functional I characterizes a "generalized distance" of the point in the space of the parameters of the object to the boundary of the regions of stability in the presence of natural and structural instability.

Let us formulate the following extremal problem: let us find the maximum in the set of parameters p_i $(i=1,2,\ldots,n)$ and the minimum of the corresponding peaks in the set of parameters q_j $(j=1,2,\ldots,m)$. This corresponds to the choice of structural parameters of the object most advantageous in the sense of the generalized distance to the boundary of the region of stability, minimization of the degree of instability and the best choice of the operating parameters (the guarantee of the calculation "in reserve"). The smallness of the deviations of $q_j{}^0$ insures the possibility of solving this problem in three steps:

Comparison of the various compositional systems of the spacecraft on the basis of analyzing the corresponding region of stability in the space of the defining parameters and the rejection of unfavorable versions;

Maximizing the functional I for the selected configuration of tanks and values of the weight coefficients λ_1 and λ_2 for the fixed

$$q_j = q_j^0 \quad (j=1, 2, \ldots, m);$$

Minimization of the maximum values of I corresponding to the optimal values found for $p_i = p_i^0$ $(i=1,2,\ldots,m)$.

Let us explain the discussed system in an example.

Assuming that the compositional system of the space vehicle is selected and the geometric configuration of the tanks is fixed at the same time, as the parameters ν_k let us take the coordinate of the center of the masses of the "dry object" x_{T0} and the coordinates of the characteristic points of the tanks, for example, the poles of the bottoms x_{01} and x_{02} of the fuel tanks, and as the parameters q_j, the densities ρ_{01}, ρ_{02} and the initial depths h_{10}, h_{20}.

The problem consists in selecting the values of these parameters giving the solution to the problem for the "minimax" of the functional I under the conditions:

$$\alpha_0 \leqslant x_{\tau 0} \leqslant \beta_0, \quad \alpha_{0i} \leqslant x_{0i} \leqslant \beta_{0i},$$
$$\varrho_{0i}^{(0)} \leqslant \varrho_i \leqslant \varrho_{0i}^{(0)} + \Delta\varrho_{0i}^{(0)}, \qquad (4.4)$$
$$h_{0i}^{(0)} \leqslant h_i \leqslant h_{0i}^{(0)} + \Delta h_{i0}^{(0)},$$

where α_0, β_0, α_{0i}, β_{0i}, $\varrho_{0i}^{(0)}$, $h_{0i}^{(0)}$, $\Delta\varrho_{0i}^{(0)}$, $\Delta h_{0i}^{(0)}$ must be considered given constants.

The most difficult is the first part of the problem inasmuch as the variations of the parameters x_{T0}, x_{01}, x_{02} are not assumed to be small. As an illustration let us consider the example of its solution for $\lambda_1=0$, $\lambda_2=1$. The dimensionless variables $Z_0=1/c$, Z_1, Z_2 corresponding to certain rated values of the parameters x^*_0, x^*_{01}, x^*_{02} and also the values of ψ as a function of the parameter are presented in Table 4.5.

Table 4.5

τ	Z_0	Z_1	Z_2	sign ψ
0	2,437	0,827	−0,868	+
0,2	2,386	0,838	−0,809	+
0,6	2,234	0,805	−0,701	−
1,0	3,109	0,645	−0,642	−

Table 4.6

τ	Z_0	Z_1	Z_2	sign D
0	2,364	0,922	−0,914	+
0,2	2,380	0,935	−0,870	+
0,6	2,229	0,907	−0,796	+
1,0	2,153	0,747	−0,760	+

Let us assign the following form to the inequalities (4.4):

$$0{,}6\overset{\bullet}{x}_{01} \leqslant x_{01} \leqslant \overset{\bullet}{x}_{01};\ \overset{\bullet}{x}_{02} \leqslant x_{02} \leqslant 1{,}2\overset{\bullet}{x}_{02};\ 0{,}8\overset{\bullet}{x}_{\tau 0} \leqslant x_{\tau 0} \leqslant 1{,}13\overset{\bullet}{x}_{\tau 0}.$$

The realization of the process of random search on a digital computer leads to the following values of parameters x_{T0}, x_{01}, x_{02}, maximizing the given functional: $x_{\tau 0}=0{,}97\overset{\bullet}{x}_{01},\ x_{02}=1{,}16\overset{\bullet}{x}_{02},\ x_{\tau 0}=0{,}87\overset{\bullet}{x}_{\tau 0}.$ The dimensionless coordinates $z_0=1/c$ corresponding to the values; Z_1, Z_2 and the functions ψ are presented in Table 4.6.

In Fig 4.5, b we have the lines $\Gamma(\tau)$ corresponding to the initial (Table 4.5) and improved (Table 4.6) versions of the object of control.

From the results of the calculations presented in Table 4.6 and in Fig 4.4, b, it follows that in the given case it is possible to make the space vehicle structurally stable in the entire range of variation of the parameter τ, increasing the distance between the fuel tanks and shifting the center of masses of the dry object.

4.3. Stabilization of Dynamically Unstable Space Vehicles (Calculation of the Damping Coefficients and the Parameters of the Automatic Stabilization System)

General Remarks. Examples

Let us consider the system of equations of disturbed motion of the space vehicle of classical design in the form

$$\begin{gathered}
(m^0+m)\ddot{z}+\lambda_1\ddot{s}_1+\lambda_2\ddot{s}_2+P\psi=b_{zu}u;\\
(J^0+J)\ddot{\psi}+\lambda_{01}\ddot{s}_1+\lambda_{02}\ddot{s}_2=b_{\psi u}u;\\
\mu_1(\ddot{s}_1+\varepsilon_1\sigma_1\dot{s}_1+\sigma_1^2 s_1)+\lambda_1\ddot{z}+\lambda_{01}\ddot{\psi}=0;\\
\mu_2(\ddot{s}_2+\varepsilon_2\sigma_2\dot{s}_2+\sigma_2^2 s_2)+\lambda_2\ddot{z}+\lambda_{02}\ddot{\psi}=0;\\
u=k_0\psi+k_1\dot{\psi};\\
k_0(\sigma)=\mathrm{Re}\,[L(i\sigma)];\ k_1(\sigma)=\mathrm{Im}\,[L(i\sigma)].
\end{gathered} \tag{4.5}$$

Let us write the characteristic equation of the system (4.5), setting $u(t)\equiv 0$ and considering smallness of the coefficients $\varepsilon_i (i=1,2)$:

$$\begin{gathered}
\Phi_\varepsilon=p^4\{[1-(\nu_1+\nu_2)+\nu_3]\,p^4+[\varepsilon_1(1-\nu_{2\zeta})+\varepsilon_2(1-\nu_{1\zeta})]\,p^3+\\
+\sigma^2[2-(\nu_1+\nu_2)+\nu_{1\zeta}+\nu_{2\zeta}]\,p^2+\sigma^2[\varepsilon_1(1+\nu_{2\zeta})+\varepsilon_2(1+\nu_{1\zeta})]\,p+\\
+\sigma^4[1+\nu_{1\zeta}+\nu_{2\zeta}]=0.
\end{gathered} \tag{4.6}$$

Let us simplify the equation (4.6), dropping the factor p^4 in equation (4.6) corresponding to the zero roots ofthe equation $\Phi_\varepsilon=0$:

$$p^4+fp^3+2ap^2+\theta fp+b=0; \tag{4.7}$$

here

$$f=\varepsilon\frac{1-\nu_2+\eta(1-\nu_1)}{1-(\nu_1+\nu_2)+\nu_3};\ \theta=\frac{1+\nu_{2\zeta}+\eta(1+\nu_{1\zeta})}{1-\nu_2+\eta(1-\nu_1)}\sigma^2;$$
$$a=\frac{2-(\nu_1+\nu_2)+\nu_{1\zeta}+\nu_{2\zeta}}{1-(\nu_1+\nu_2)+\nu_3};\ b=\frac{1+\nu_{1\zeta}+\nu_{2\zeta}}{1-(\nu_1+\nu_2)+\nu_3};\tag{4.8}$$
$$\eta=\frac{\varepsilon_2}{\varepsilon_1};\ \varepsilon_1=\varepsilon;$$
$$\nu_1=\frac{\lambda_1^2}{\mu_1(m^0+m)}+\frac{\lambda_{01}^2}{\mu_1(J^0+J)};\ \nu_2=\frac{\lambda_2^2}{\mu_2(m^0+m)}+\frac{\lambda_{02}^2}{\mu_2(J^0+J)};$$
$$\nu_{1\zeta}=\frac{\lambda_{01}\lambda_1}{\mu_1(J^0+J)(m^0+m)}\left(\frac{P}{\sigma^2}\right);\ \nu_{2\zeta}=\frac{\lambda_{02}\lambda_2}{\mu_2(m^0+m)(J^0+J)}\left(\frac{P}{\sigma^2}\right);$$
$$\nu_3=\frac{(\lambda_{01}\lambda_2-\lambda_{02}\lambda_1)^2}{\mu_1\mu_2(m^0+m)(J^0+J)}.$$

Let us begin with the examples illustrating the characteristic influence on the stability of the spacecraft of the dissipative forces caused by the oscillations of the fuel in the compartments of the space vehicle. Let us take the following numerical values for the coefficients of the system (4.5).

A. Space vehicle with cylindrical fuel compartments:

$$\begin{aligned}
&\frac{\lambda_1}{\mu_1}=-0{,}0424; && \frac{\lambda_1}{m^0+m}=-1{,}541; && \frac{b_{zu}}{m^0+m}=-24{,}31;\\
&\frac{\lambda_2}{\mu_2}=-0{,}0231; && \frac{\lambda_2}{m^0+m}=-1{,}541; && \frac{b_{\psi u}}{J^0+J}=-14{,}75;\\
&\frac{\lambda_{01}}{\mu_1}=0{,}0115; && \frac{\lambda_{01}}{J^0+J}=4{,}686; && \frac{P}{m^0+m}=-37{,}64;\\
&\frac{\lambda_{02}}{\mu_2}=-0{,}00825; && \frac{\lambda_{02}}{J^0+J}=-6{,}164; && \sigma_1^2=\sigma_2^2.
\end{aligned}\tag{4.9}$$

B. Space vehicle with toroidal fuel compartments:

$$\begin{aligned}
&\frac{\lambda_1}{\mu_1}=-0{,}884; && \frac{\lambda_1}{m^0+m}=-0{,}305; && \frac{b_{zu}}{m^0+m}=-0{,}363;\\
&\frac{\lambda_2}{\mu_2}=-0{,}808; && \frac{\lambda_2}{m^0+m}=-0{,}193; && \frac{b_{\psi u}}{J^0+J}=-1{,}680;\\
&\frac{\lambda_{01}}{\mu_1}=0{,}369; && \frac{\lambda_{01}}{J^0+J}=0{,}407; && \frac{P}{m^0+m}=-20{,}0;\\
&\frac{\lambda_{02}}{\mu_2}=0{,}065; && \frac{\lambda_{02}}{J^0+J}=0{,}611, && \sigma_1^2=15{,}20;\\
& && && \sigma_2^2=13{,}90.
\end{aligned}\tag{4.10}$$

The roots $p_j=\alpha_j+i\omega_j$ of the characteristic equation (4.6) close to the partial frequencies of the fuel oscillations σ_1, σ_2 calculated for the various decrements of the oscillations

$$\delta_1 = \frac{\varepsilon_1 \pi}{\sigma_1}; \quad \delta_2 = \frac{\varepsilon_2 \pi}{\sigma_2}$$

are presented in the tables 4.7 and 4.8 for examples A and B, respectively.

Table 4.7

δ_1	δ_2	$p_1, \bar{p}_1$	$p_2, \bar{p}_2$
0	0	−0,0385±i7,1079	0,0385±i7,1079
0,002	0,05	−0,0796±i7,0963	0,0184±i7,1194
0,05	0,002	−0,0814±i7,1196	0,0182±i7,0961
0,05	0,05	−0,0983±i7,1083	−0,0217±i7,1071
0	0,1	−0,1303±i7,0807	0,0133±i7,1242
0,1	0,1	−0,1582±i7 1082	−0,0809±i7,1057
0,1	0	20,1349±i7,1236	0,0130±i7,0913

Table 4.8

δ_1	δ_2	$p_1, \bar{p}_1$	$p_2, \bar{p}_2$
0	0	−0,412±i3,229	0,413±i3,229
0	0,25	−0,530±i3,281	0,330±i3,171
0	0,50	−0,674±i3,313	0,275±i3,122
0	1,0	−1,005±i3,310	0,208±i3,056
0,25	0	−0,672±i2,978	0,318±i3,456
0,50	0	−0,984±i2,784	0,276±i3,578
1,0	0	−1,628±i2,358	0,213±i3,692
0,2	0,2	−0,667±i3,050	0,224±i3,386
0,3	0,3	−0,809±i2,966	0,145±i3,744
0,6	0,6	−1,248±i2,707	−0,080±i3,554

The calculation data make it possible to note the following.

1. In the absence of controlling $[u(t)\equiv 0]$ and dissipative forces $[\varepsilon_1=\varepsilon_2=0]$ the space vehicle is dynamically unstable -- the roots of the equation (4.6) have positive real parts, $p=0.0385\pm i7.1079$ (version A), $p=0.4120\pm i3.2290$ (version B).

2. The stabilization of the space vehicle is not insured by damping the fuel oscillations in any one of the compartments even with a significant increase in the corresponding decrements to the values of $\delta=0.10$ (version A), $\delta=1.00$ (version B).

3. The symmetric damping of the oscillations in each of the compartments permits us to insure stability of the space vehicle for comparatively small decrements of the oscillations of the fuel components: $\delta_1=\delta_2=0.05$ (for version A), $\delta_1=\delta_2=0.60$ (for version B).

Thus, the preliminary analysis of the effect on the stability of the spacecraft of the dissipative forces leads to the results greatly resembling the conclusions obtained for a model of a double pendulum investigated in Chapter 1. Therefore the problem of the stabilization of the dynamics of the unstable spacecraft needs more detailed investigated, just as the mentioned model, also calling on the control system and stabilizing factor (the automatic angular stabilization system).

Investigation of the Regions of Stability

First let us consider the problem of the effect of the dissipative forces on the region of dynamic instability of the uncontrollable space vehicle (in the Z_1, Z_2 plane).

Let the following load be such that the system (4.5) will be at the stabilitary boundary for $\varepsilon_1=\varepsilon_2=0$, $u(t)\equiv 0$ (the existence of this boundary has been established in the preceding section). Here $b=a^2$ and the characteristic equation

$$p^4+2ap^2+b=0 \qquad (4.9a)$$

has a multiple root $p^2=-a$.

Let us investigate the behavior of the solutions of equations (4.9) near the investigated stability boundary. Since all of the coefficients of the equation (4.9) are positive, from the four Hurwitz inequalities

$$\Delta_1=f>0;\ \Delta_2=f(2a-\theta)>0;$$
$$\Delta_3=f^2(2a\theta-b-\theta^2);\ \Delta_4=f^2b(2a\theta-b-\theta^2) \qquad (4.10a)$$

only the inequality

$$\Delta_3=f^2(2a\theta-b-\theta^2)>0. \qquad (4.11)$$

is nontrivial.

It is easy to see that for $b=a^2$ the inequality (4.10) is not satisfied:

$$\Delta_3=-f^2(\theta-a)^2\leqslant 0 \qquad (4.12)$$

(the case $\theta=a$ will be considered special). As is obvious from the expression (4.10):

$$\Delta_1>0;\ \frac{\Delta_2}{\Delta_1}>0;\ \frac{\Delta_3}{\Delta_2}<0;\ \frac{\Delta_4}{\Delta_3}>0,$$

therefore the characteristic equation (4.9) has two roots with positive real parts Rep. It is easy to find these roots in the vicinity of the value of $\theta=a$. Let us set $\theta=a(1+\delta)$, where δ is small. Then with accuracy to the terms of second order of smallness with respect to δ:

$$p_{1,2}=\frac{a}{2f}\delta^2\pm i\sqrt{a}\left(1+\frac{1}{2}\delta-\frac{1}{8}\delta^2\right); \qquad (4.13)$$

the other two roots:

$$p_{3,4} = -\frac{1}{2}\left(f + \frac{a\delta^2}{f}\right) \pm i\sqrt{a}\left(1 - \frac{1}{2}\delta + \frac{1}{8}\delta^2\right).$$

In the case of $\theta=a$ ($\delta=0$) we have:

$$p_{1,2} = \pm i\sqrt{a};\; p_{3,4} = -\frac{1}{2} f \pm i\sqrt{a}. \qquad (4.14)$$

Thus, from the expressions (4.12)-(4.14) it follows that the introduction of the small partial damping into the investigated nonconservative system on the stability boundary either converts it to an unstable system or leaves it on the stability boundary. The result is determined by the relation between the "distribution of the damping resistances" θ and the difference in the natural frequencies of the system (4.5) characterized by the value of $x=\sqrt{1-b/a^2}$.

The conclusion drawn is confirmed by direct calculation of the regions of instability in the plane of the parameters Z_1, Z_2. The stability boundary of the system (4.5) in the investigated case is given by the equation

$$\Phi_\eta = \theta^2 - 2a\theta + b = 0$$

or in the notation (4.B)

$$\begin{gathered}\Phi_\eta = -[1-(\nu_1-\nu_2)+\nu_3][1+\nu_{2\zeta}+\eta(1+\nu_{1\zeta})]^2 + \\ +[1-\nu_2+\eta(1-\nu_1)][2-(\nu_1+\nu_2)+\nu_{1\zeta}+\nu_{2\zeta}][1+\nu_{2\zeta}+\eta(1+\nu_{1\zeta})] - \\ -[1-(\nu_{1\zeta}+\nu_{2\zeta})][1-\nu_2+\eta(1-\nu_1)]^2 = 0. \qquad (4.15)\end{gathered}$$

The system (4.5) is stable if $\Phi_\eta>0$ and unstable if $\Phi_\eta<0$. In Fig 4.6 the curves (4.15) separating the regions of stability and the regions of instability are constructed for different values of the parameter η for fixed values of the parameters k, γ and ζ. As is obvious from Fig 4.6 and the results of the numerical analysis, the case $\eta=1$ (partial damping is identical in both compartments) is the most favorable, for in this case the regions of instability are small (crosshatched in Fig 4.6). Let us consider this case in more detail. For $\eta=1$ the equation (4.15) using dimensionless parameters (3.52) is reduced to the form

$$\begin{gathered}\Phi_{\eta=1} = \{2\sqrt{k}(Z_2-Z_1)[1+\zeta\gamma(Z_1+kZ_2)] - [Z_1^2+kZ_2^2+(1+k)+ \\ +2\zeta(Z_1+kZ_2)]\}\{2\sqrt{k}(Z_2-Z_1)[1+\zeta\gamma(Z_1+kZ_2)]+ \\ +[Z_1^2+kZ_2^2+(1+k)+2\zeta(Z_1+kZ_2)]\} = 0. \qquad (4.16)\end{gathered}$$

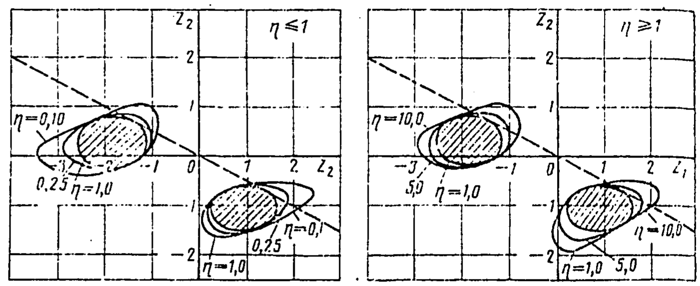

Figure 4.6. The nature of the effect of the damping of the fuel oscillations in the compartments ($\varepsilon_1 \ll 1$, $\varepsilon_2 \ll 1$) on the regions of dynamic instability of the space vehicle

Equation (4.16) defines the ellipses with the centers at the points (ξ_1, ζ_1), (ξ_2, ζ_2) and the principal axes rotated by the angles α_1 and α_2, respectively in the plane Z_1, Z_2. The coordinates ξ_i, ζ_i, α_i (i=1,2) are defined by the formulas

$$\xi_{1,2}=\frac{\mp\sqrt{k}-\zeta+B\zeta\pm B\sqrt{k}}{1-(B\zeta)^2};\quad \zeta_{1,2}=\frac{\mp k^{-1/2}-\zeta+B\zeta\pm B\zeta^2k^{-1/2}}{1-(B\zeta)^2};$$
$$\alpha_{1,2}=\frac{1}{2}\operatorname{arctg}\frac{2B\zeta(k-1)(k+1)^{-1}}{2B\zeta\mp(k-1)k^{-1/2}};$$
$$B=\gamma(1+k)$$

(here the plus and minus signs are in accordance with the indexes 1 and 2).

Consideration of the term $\varepsilon_1\varepsilon_2$ which we have neglected above leads to a decrease in the regions of natural dynamic instability (see Fig 4.7). From a comparison of Figures 4.6 and 4.7 it follows that the damping of the liquid oscillations in two compartments is simultaneously a necessary factor stabilizing the system. The additional substantiation of this fact is presented later.

Let us note that since

$$\Delta\Phi=\Phi_\eta-\Phi_{\eta-1}=4k\gamma^4\zeta^2(Z_2-Z_1)^2(Z_1+kZ_2)^2\geqslant 0,$$

the regions of natural dynamic instability of the system (4.5) for different partial damping ($\eta\neq 1$) include the regions of instability for the case $\eta=1$; in turn, the latter include the regions of instability obtained for the system without dissipation. The above-investigated special case of $\theta=a$ corresponds to the common points of the boundaries of the regions $\Phi_\eta=0$ and $\Phi_{\eta-1}=0$.

The problem of the asymptotic behavior of the boundary of the regions of instability (4.15) is of interest for the various laws under which ε_1 and ε_2 approach zero:

a) Let $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, η=const.

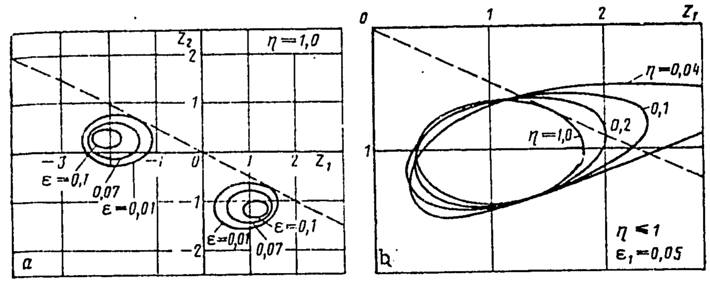

Figure 4.7. Nature of the effect of large damping on the region of dynamic instability of the space vehicle

As is easy to see, this case corresponds to the initial expression (4.15);

b) Let $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\eta \to 0$.

From expression (4.15) we obtain

$$\Phi_{\eta\to 0}=\nu_3(1+\nu_{2\zeta})^2-(\nu_{2\zeta}+\nu_2)[\nu_1+\nu_{1\zeta}+\nu_1\nu_{2\zeta}-\nu_2\nu_{1\zeta}]=0; \quad (4.17)$$

c) Let, finally, $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\eta \to \infty$.

After certain transformations from equation (4.15) we obtain

$$\Phi_{\eta\to\infty}=\nu_3(1+\nu_{1\zeta})^2-(\nu_{1\zeta}+\nu_1)[\nu_2+\nu_{2\zeta}+\nu_2\nu_{1\zeta}-\nu_1\nu_{2\zeta}]=0. \quad (4.18)$$

Comparing expressions (4.15), (4.16), and (4.17), it is possible to see that in all three limiting cases the boundaries of the region of stability obtained differ significantly and they differ from the corresponding boundaries $\Phi=0$ obtained under the assumption that the dissipative forces are identically equal to zero.

In the general case $\varepsilon_1 \neq \varepsilon_2$ the condition of stability of the system (4.5) has the form

$$\begin{aligned}\Phi_\eta=&[1-(\nu_1+\nu_2)+\nu_3][1+\nu_{2\zeta}+\eta(1+\nu_{1\zeta})+\beta k\gamma]^2-\\ &-[1-\nu_2+\eta(1-\nu_1)][2-(\nu_1+\nu_2)+\nu_{1\zeta}+\nu_{2\zeta}+\beta k\gamma(1-\nu_1)]\times\\ &\times[1+\nu_{2\zeta}+\eta(1+\nu_{1\zeta})+\beta k\gamma]+[1+\nu_{1\zeta}+\nu_{2\zeta}+\beta k\gamma(1+\nu_{1\zeta})]\times\\ &\times[1-\nu_2+\eta(1-\nu_1)]<0. \quad (4.19)\end{aligned}$$

The numerical analysis indicates that the curves $\Phi_\eta=0$ in the plane Z_1, Z_2 also have the form presented in Fig 4.6 and 4.7. When $\beta=0$ they become the corresponding curves (4.15). For the general case we have all the conclusions obtained above.

Let us note the important difference of the above-investigated problems from the corresponding nonconservative problems of the theory of elastic stability consisting in the fact that the former cannot be formulated as problems of finding the critical value of the parameter (in the given case the values of $j=p/(m^0+m)$. Actually, the value of j, as is obvious from expressions (4.8) enters into the equalities characterizing the stability of the system only in combination with the square of the natural frequency $j/2\sigma^2$. The latter ratio denoted above as ζ, in the sense that the problem does not depend on the value of this parameter. The dynamic properties of the investigated system are defined by the entire set of parameters Z_1, Z_2, k, γ, ζ, β.

Thus, the investigated form of the dynamic instability of the object of control has an entire series of peculiarities which require known caution when developing recommendations to insure stability of a closed system; in some cases on introduction of dissipative forces the effect can turn out to be opposite to what is desired. This problem requires further analysis with consideration of the effect of the control system.

Calculation of the Damping Coefficient

The characteristic equation of system (4.5), omitting the factor p^4, is represented in the form

$$F(p)=F_0(p^2)+pF_1(p^2), \qquad (4.20)$$

where

$$F_0(p^2)=\Phi_0+k_0\Psi_0+\varepsilon_1\varepsilon_2\sigma_1\sigma_2\Phi_{12}+k_1\varepsilon_1\sigma_1p^2\Phi_1+k_1\varepsilon_2\sigma_2^2p^2\Psi_{12}+ + k_0\varepsilon_1\varepsilon_2\sigma_1^2\sigma_2^2\Psi_{12};$$

$$F_1(p^2)=\varepsilon_1\sigma_1\Phi_1+\varepsilon_2\sigma_2\Phi_2+k_1\Psi_0+k_0\varepsilon_1\sigma_1\Psi_1+ + k_0\varepsilon_2\sigma_2\Psi_2+k_1\varepsilon_1\varepsilon_2\sigma_1\sigma_2p^2.$$

Then for the purpose of some simplification we set $\sigma_1=\sigma_2=\sigma$ and we introduce the new variable $x=(p/\sigma)^2$. Expanding the determinants Φ_3, Ψ_0, Φ_1, Ψ_1, Φ_2, Ψ_2, Φ_{12}, Ψ_{12}, we reduce the expressions for F_0, F_1 to the form

$$F_0(x)+x(a_0x^2+a_1x+a_2)+\varrho_0(b_0x^2+b_1x+b_2)+ + \varepsilon_1\varepsilon_2(x^2+\varrho_0x)+\varepsilon_1\varrho_1x(c_0x+1)+\varepsilon_2\varrho_1x(d_0x+1); \qquad (4.21)$$

$$F_1(x)=\varrho_1(b_0x^2+b_1x+b_2)+\varepsilon_1x(f_1x+f_2)+\varepsilon_2x(g_1x+g_2)+ + \varepsilon_1\varrho_0x(c_0x+1)+\varepsilon_2\varrho_0x(d_0x+1)+\varepsilon_1\varepsilon_2\varrho_1x. \qquad (4.22)$$

The coefficients of the polynomials F_0, F_1 are positive and are related to the coefficients of the initial system (4.5) by the following expression:

$$a_0=1-(\nu_1+\nu_2)+\nu_3; \qquad b_0=1-(\nu_4+\nu_5);$$
$$a_1=2-(\nu_1+\nu_2)+(\nu_{1\zeta}+\nu_{2\zeta}); \quad b_1=2-(\nu_4+\nu_5);$$
$$a_2=1+\nu_{1\zeta}+\nu_{2\zeta}; \qquad b_2=1;$$
$$d_0=1-\nu_4;\ f_1=1-\nu_2;\quad g_1=1-\nu_1$$
$$c_0=1-\nu_5;\ f_2=1+\nu_{2\zeta};\quad g_2=1+\nu_{1\zeta};$$
$$\nu_1=\frac{\lambda_1^2}{\mu_1(m^0+m)}+\frac{\lambda_{01}^2}{\mu_1(J^0+J)};\ \nu_2=\frac{\lambda_2^2}{\mu_2(m^0+m)}+\frac{\lambda_{02}^2}{\mu_2(J^0+J)};$$
$$\nu_3=\frac{(\lambda_{01}\lambda_2-\lambda_{02}\lambda_1)^2}{\mu_1\mu_2(m^0+m)(J^0+J)};\ \nu_4=\frac{\lambda_1^2}{\mu_1(m^0+m)}+\frac{\lambda_1\lambda_{01}}{\mu_1(m^0+m)}\left(\frac{b_{zu}}{b_{\psi u}}\right);$$
$$\nu_5=\frac{\lambda_2^2}{\mu_1(m^0+m)}+\frac{\lambda_2\lambda_{02}}{\mu_2(m^0+m)}\left(\frac{b_{zu}}{b_{\psi u}}\right);$$
$$\nu_{1\zeta}=\frac{\lambda_{01}\lambda_1}{\mu_1(J^0+J)(m^0+m)}\left(\frac{P}{\sigma^2}\right);\ \nu_{2\zeta}=\frac{\lambda_{02}\lambda_2}{\mu_2(m^0+m)(J^0+J)}\left(\frac{P}{\sigma^2}\right);$$
$$\varrho_0=-\frac{k_0a_{\psi u}}{\sigma^2};\ \varrho_1=-\frac{k_1a_{\psi u}}{\sigma^2}.$$

Let us assume that the equation (4.20) for $F_1=0$

$$\Phi_0(p^2)=p^4(a_0p^4+a_1p^2+a_2)=0 \qquad (4.23)$$

has, in addition to the zero roots, two pairs of complex-conjugate roots

$$p=\pm\alpha\pm i\omega, \qquad (4.24)$$

which is possible if the condition is satisfied:

$$a_1^2-4a_0a_2<0 \qquad (4.25)$$

(the condition of dynamic instability of the object of control).

The values of ε_1, ε_2, ρ_0, ρ_1 are free parameters which must be available so that the polynomial will be Hurwitz. For this purpose, on the basis of the Hermite-Bealer theorem, it is necessary and sufficient:

a) That the higher-order coefficients of the polynomials $F_0(x)$ and $F_1(x)$ have identical signs ($x=(p/\sigma)^2$);

b) That the roots u_i, v_i of the polynomials $F_0(x)$, $F_1(x)$ be different, real and negative;

c) That they be permutated as follows:

$$u_1<v_1<u_2<\ldots<0.$$

The condition "a" for equation (4.20) is satisfied automatically as a consequence of the positive determinacy of the quadratic form corresponding to the kinetic energy of the space vehicle and also other properties of the system (4.5).

Further study of equation (4.20) is broken down into two steps:

Analysis of the conditions corresponding to reality and negativeness of the roots of the equation $F_0=0$, $F_1=0$ (the condition of dynamic stability of the object of control);

Analysis of the conditions which must be satisfied by the parameters ε, ρ, in order to insure the necessary permutatability of the roots of the equations $F_0=0$; $F_1=0$.

Let us set $\rho_0=\rho_1=0$ in the expressions (4.21), (4.22), and let us consider the equation

$$F_0(x)=a_0x^2+a_1x+a_2+\nu x=0; \qquad (4.26)$$

$$F_1(x)=(f_1x+f_2)+\eta(g_1x+g_2)=0, \qquad (4.27)$$

where $\nu=\varepsilon_1\varepsilon_2>0;\ \eta=\frac{\varepsilon_2}{\varepsilon_1}>0.$

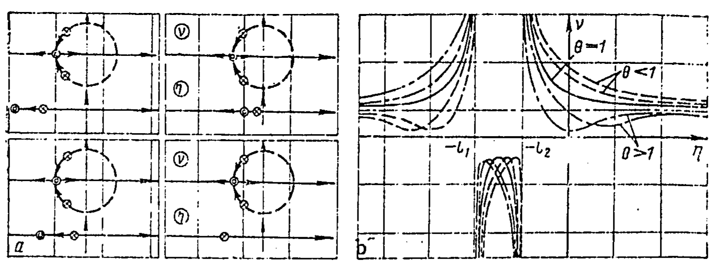

Figure 4.8. Nature of the effect of the damping of the fuel oscillations on the stability of the space vehicle (a) and the form of the function ν(η) at the dynamic instability boundary of the space vehicle (b)

For investigation of the roots of equations (4.26), (4.27) we shall use the root hodograph method.

Let us first return to equation (4.26). The equation of the root hodograph in the plane μ=Re x, Ω=Im x

$$\Omega^2+\mu^2=R^2$$

is the equation of a circle of radius $R=\sqrt{a_2/a_0}$ with its center at the origin of the coordinates (Fig 4.8, a). Here and hereafter the solid lines correspond to positive values of the parameter ν, and the dotted lines, the negative values of this parameter. Obviously the values of $\nu>0$ have physical meaning.

For $\nu_*=2\sqrt{a_0a_2}-a_1$ the roots of the equation (4.26) become real. Thus, in order to satisfy the condition "b" the parameter $\nu=\varepsilon_1\varepsilon_2$ must be selected so that the inequality $\nu>\nu_*=-a_1+2\sqrt{a_0a_2}$ will be satisfied.

Let us note that the magnitude of ν_* is positive, for the inequality

$$-a_1+2\sqrt{a_0a_2}>0$$

is a consequence of the condition (4.25) of the natural dynamic instability of the object which, in accordance with the proposition, is satisfied.

Let us thus consider the equation (4.27). From the properties of the system (4.5) we have directly:

$$f_1>0;\ f_2>0;\ g_1>0,\ g_2>0.$$

The root hodographs corresponding to this equation are therefore segments of the real axis in the plane (μ, Ω).

Some of the possible situations in the arrangement of the initial ($\eta=0$) and limiting ($\eta\to\infty$) points, including comparison of them are presented in Fig 4.8, a. The arrows indicate the direction of displacement of the roots with increase in the parameter $\eta>0$.

Comparing the hodographs corresponding to the parameters ν,η (located under each other in Fig 4.8, a), we note that in all cases there are a pair of numbers (ν, η) such that the root of the equation (4.26) coincides with one of the roots of the equation (4.27); therefore, with a further increase in one of the parameters ν, η the condition "v" will be satisfied without violation of the condition "b."

As is obvious, the stabilization of the dynamically unstable space vehicle is impossible if the damping of the fuel oscillations is realized only in one of the compartments ($\varepsilon_1=0$ or $\varepsilon_2=0$).

It is possible to realize stabilization of the space vehicle by selecting the characteristics of the damping devices in such a way as to insure the required values of the parameter $\nu=\varepsilon_1\varepsilon_2$ and the parameter $\eta=\varepsilon_2/\varepsilon_1$ considering possible technological restrictions.

The problem of selecting the required relation between ε_1 and ε_2 is of practical interest. Let us consider in more detail, using the properties of the initial system 4.5 to a greater degree.

The condition "v" of permutatability of the roots of equations (4.26), (4.27) can be obtained directly. Actually, let

$$\xi = -\frac{f_2 + \eta g_1}{f_1 + \eta g_2}$$

be the root of equation (4.27). Then, calculating $F_0(\xi)$, we obtain the required inequality

$$F_0(\xi) = a_0\xi^2 + (a_1 + \nu)\xi + a_2 < 0. \qquad (4.28)$$

The equation

$$a_0\xi^2 + (a_1 + \nu)\xi + a_2 = 0 \qquad (4.29)$$

gives the required relation between the parameters ν, η at the limit of stability of the system (4.5). Then the parameters ν, η satisfying the equation (4.29) will be denoted as $\nu=\nu_*$; $\eta=\eta_*$.

Let us introduce the notation:

$$l_1 = \frac{f_1}{g_1}; \quad l_2 = \frac{f_2}{g_2}; \quad g = \frac{g_2}{g_1}; \quad \theta = \frac{a_0 g^2}{a_2}.$$

Then equation (4.29) can be written in the form

$$\nu_{(\eta)} = \frac{a_2}{g}\left(\theta\frac{l_2 + \eta}{l_1 + \eta} + \frac{l_1 + \eta}{l_2 + \eta} - \frac{a_1}{a_2} g\right). \qquad (4.30)$$

The problem of stabilization of the space vehicle in the oscillation frequency band of the fuel in the compartments thus consists in selecting the damping coefficients ε_1, ε_2 from the given region $0<\varepsilon_1\leq\varepsilon_{1m}$; $0<\varepsilon_2\leq\varepsilon_{2m}$ defined by the inequality (4.28). The choice of smaller values of the parameters ε_1, ε_2 is preferable.

Let us investigate equation (4.30) in more detail, including, in addition to the parameters $\nu=\varepsilon_1\varepsilon_2$, $\eta=\varepsilon_2/\varepsilon_1$ which characterize the dissipative forces occurring during the oscillations of the fuel in the space vehicle compartments, also the parameters

$$\theta = \frac{a_0}{a_2} g^2,\ a_1,\ a_2,\ l_1,\ l_2,\ g,$$

directly determined by its compositional layout.

The function $\nu(\eta)$ will be represented in the form

$$\nu(\eta) = \frac{a_0 g^2 (l_2 + \eta)^2 - a_1 g (l_2 + \eta) + a_2 (l_1 + \eta)^2}{g (l_2 + \eta)(l_1 + \eta)}. \qquad (4.31)$$

Calculating the discriminant of the quadratic trinomial (with respect to η) in the numerator of the fraction (4.31), we find

$$D = g^2 (l_1 - l_2)^2 (a_1^2 - 4a_0 a_2),$$

obviously, on the basis of the condition (4.25), D<0.

This means that the function $\nu(\eta)$ does not have real zeros. The ones of this function are determined from the relations

$$\eta_1^0 = -l_1 = \frac{f_1}{g};\ \eta_2^0 = -l_2 = \frac{f_2}{g}.$$

Calculating the derivatives of the function $\nu(\eta)$, we find

$$\frac{d\nu}{d\eta} = \frac{a_2(l_1-l_2)}{g} \frac{(\theta-1)\eta^2 + 2(\theta l_2 - l_1)\eta + (\theta l_2^2 - l_1^2)}{(\eta+l_1)^2(\eta+l_2)^2}; \quad (4.32)$$

$$\frac{d^2\nu}{d\eta^2} = -\frac{a_2(l_1-l_2)}{g} \frac{(\theta-1)\eta^3 + 3(\theta l_2 - l_1)\eta^2 + 3(\theta l_2^2 - l_1^2)\eta + (\theta l_2^3 - l_1^3)}{(\eta+l_1)^2(\eta+l_2)^2}. \quad (4.33)$$

From expressions (4.31)-(4.33) we find the points of the extremum (for $\theta \neq 1$):

$$\eta_1 = \frac{l_1 - \sqrt{\theta} l_2}{\sqrt{\theta} - 1};\ \eta_2 = -\frac{l_1 + \sqrt{\theta} l_2}{\sqrt{\theta} + 1},$$

and also the inflection point of the function $\nu(\eta)$:

$$\eta_n = \frac{l_1 - \sqrt[3]{\theta} l_2}{\sqrt[3]{\theta} - 1}.$$

Calculating the asymptote of the function $\nu(\eta)$ (for $\eta \to \pm\infty$)

$$\nu_a = \frac{a_2}{g}\left(\theta - \frac{a_1}{a_2} g + 1\right),$$

as the characteristic point (for $\theta \neq 1$) we also find

$$\eta_0 = \frac{l_1 - \theta l_2}{\theta - 1}$$

-- the point where the function $\nu(\eta)$ intersects its asymptote.

The following sequence of the characteristic points of the function $\nu(\eta)$ arises:

— for $l_1 > l_2$ $\theta < 1$ (or $l_1 < l_2$, $\theta > 1$),

$$\frac{l_1 - \sqrt{\theta} l_2}{\sqrt[3]{\theta} - 1} < \frac{l_1 - \sqrt{\theta} l_2}{\sqrt{\theta} - 1} < \frac{l_1 - \theta l_2}{\theta - 1} < -\frac{l_1 + \sqrt{\theta} l_2}{\sqrt{\theta} + 1},$$

— for $l_1 > l_2$ $\theta > 1$ (or $l_1 < l_2$; $\theta < 1$),

$$-\frac{l_1 + \sqrt{\theta} l_2}{\sqrt{\theta} + 1} < \frac{l_1 - \theta l_2}{\theta - 1} < \frac{l_1 - \sqrt{\theta} l_2}{\sqrt{\theta} - 1} < \frac{l_1 - \sqrt[3]{\theta} l_2}{\sqrt[3]{\theta} - 1}$$

(Fig 4.9, a, b and 4.9, c, d, respectively).

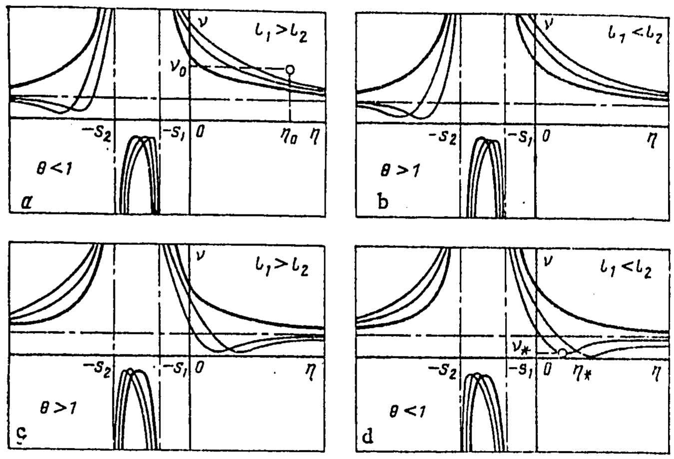

Figure 4.9. Form of the function $\nu(\eta)$ for various values of the characteristic parameters of a space vehicle

Considering that the regions $\nu>0$, $\eta>0$ in the plane (ν, η) only have physical meaning, let us consider the various cases successively:

$$1.\ l_1 > l_2;\ \theta < 1\ (l_1 < l_2,\ \theta > 1).$$

As is obvious, for $\nu \leq \nu_a$, where

$$\nu_a = \frac{a_2}{g}\left(\theta - \frac{a_1}{a_2} g + 1\right),$$

the stabilization of the space vehicle is impossible for any values of the parameters ν, η. For $\nu>\nu_a$ for each $\nu=\nu_*$ there is only one value of $\eta=\eta_0>0$, for which the point (ν_0, η_0) belongs to the region of stability of the system (4.5) (see Fig 4.9, a).

The values of ν_0, η_0 stabilizing the system will be found most simply in inverse order: being given the arbitrary value of $\eta_*>0$, let us select the values of $\nu_0=\nu(\eta_0)$ in accordance with the inequality

$$\nu_0 > \frac{a_2}{g}\left[\theta \frac{l_2+\eta_0}{l_1+\eta_0} + \frac{l_1+\eta_0}{l_2+\eta_0} - \frac{a_1}{a_2} g\right].$$

$$2.\ l_1 > l_2,\ \theta > 1\ \ (l_1 < l_2,\ \theta < 1).$$

In the given case there are minimum values of the parameters ν_*, η_*;

$$\nu_* = 2\sqrt{a_0 a_2} - a_1 > 0;\ \eta_* = \frac{l_1 - \sqrt{\theta}\, l_2}{\sqrt{\theta} - 1} > 0,$$

such that for $\eta_0=\eta_*$; $\nu_0=\nu_*+\nu_\varepsilon$ (ν_ε is a small number), the stabilization of the initial system (4.5) is insured.

If it is necessary to insure the necessary stability margins of safety, then it is expedient by selecting $\eta_0 \approx \eta_* = \frac{l_1 - \sqrt{Q} l_2}{\sqrt{Q} - 1}$ to select ν_0 beginning with the inequality

$$\nu_0 > \frac{a_2}{g}\left[\theta \frac{l_2+\eta_0}{l_1+\eta_0} + \frac{l_1+\eta_0}{l_2+\eta_0} - \frac{a_1}{a_2} g\right].$$

Let us note several special cases

3. $\theta=1;\ l_1 \neq l_2$:

$$\nu(\eta) = \frac{a_2}{g}\left[\frac{l_2+\eta}{l_1+\eta} + \frac{l_1+\eta}{l_2+\eta} - \frac{a_1}{a_2} g\right].$$

This case is intermediate between cases 1 and 2 [one maximum of the function $\nu(\eta)$ is degenerate]. However, from the point of view of selecting the parameters ν, η required for stability of the system (4.5), it does not differ from case 1 (Fig 4.10, a).

4. $\theta l_2 = l_1$;

$$\nu(\eta) = \frac{a_2}{g}\left[\theta \frac{l_2+\eta}{\theta l_2+\eta} + \frac{\theta l_2+\eta}{l_2+\eta} - \frac{a_1}{a_2}\right].$$

The points of intersection of the ν axis of the function $\nu(\eta)$ and its asymptote coincide:

$$\nu_0 = \nu_a = \frac{a_2}{g}\left[\theta + 1 - \frac{a_1}{a_2}\right].$$

The choice of the stabilizing parameters ν, η does not in the given case differ from case 2 (Fig 4.10, b).

Let us note that in Fig 4.10, a-c the crosshatching denotes the regions of instability of the system (4.5)

5. $l_1 = l_2$ $\nu = \text{const}$.

The function $\nu(\eta)$ coincides with its asymptote (Fig 4.10, c):

$$\nu = \nu_a = \frac{a_2}{g}\left(\theta - \frac{a_1}{a_2} g + 1\right).$$

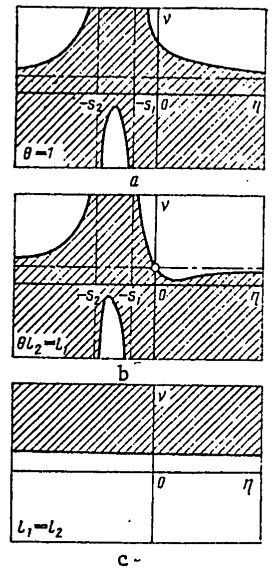

Figure 4.10. Regions of stability (1) and instability (2) of a space vehicle for various values of the characteristic parameters

Selecting $\nu_0=\nu_a+\nu_\varepsilon>0$, we find that the stability of the system (4.5) is insured for any values of $\nu=\varepsilon_2/\varepsilon_1$. The value of $\nu_\varepsilon>0$ can serve as a measure of the safety margin of stability.

The initial values of ε_1, ε_2 (dissipative coefficients) are obviously selected under the condition

$$\varepsilon_1\varepsilon_2=\text{const}=\nu_0.$$

The relation between $\varepsilon 1$, $\varepsilon 2$ can be arbitrary.

As is obvious, there are bifurcation values of the parameters $\ell=\ell_2=\ell_*$ and $\theta_*=1$ for which the qualitative form of the function $\nu(\eta)$ varies.

Let us construct the boundaries of the regions $\ell_1=\ell_2$, $\theta=1$ in the plane of the parameters

$$Z_1=\sqrt{\frac{m^0+m}{J^0+J}\frac{\lambda_{01}}{\lambda_1}};\quad Z_2=\sqrt{\frac{m^0+m}{J^0+J}\frac{\lambda_{02}}{\lambda_2}}, \qquad (4.34)$$

that is, the characteristic parameters of the space vehicle which were used to construct the regions of its stabilizability.

Let us note that the equation $\ell_1=\ell_2$, if we proceed to the system of parameters Z_1, Z_2, k, γ, ζ, differs from the equation $\theta=1$ by the presence of a term proportional to the parameter $\gamma^2=(\lambda_1/m^0+m)^2$, which plays the role of a small parameter for the investigated class of objects. Setting $\gamma\simeq 0$, the equation $\ell_1=\ell_2$ (θ=1) will be converted to the form

$$(Z_1+\zeta)^2-k(Z_2+\zeta)^2=(1-\zeta^2)(k-1), \tag{4.35}$$

where the parameters k, ζ are defined by the expressions (3.52).

The equation (4.35) is the equation of a hyperbola with its center at the point $(-\zeta, -\zeta)$, which for k=1 degenerates into the pair of intersecting straight lines. The mutual arrangement of the boundaries of the regions of natural dynamic instability

$$\frac{[Z_1+(\zeta\mp\sqrt{k})]^2}{(\zeta\sqrt{1+k})^2}+\frac{\left[Z_2+\left(\zeta\pm\frac{1}{\sqrt{k}}\right)\right]^2}{\left(\zeta\sqrt{\frac{1+k}{k}}\right)^2}=1$$

is represented in Fig 4.11, where Fig 4.11, a corresponds to the case of k<1, Fig 4.11, c, the case k>1, Fig 4.11, b, the case k=1.

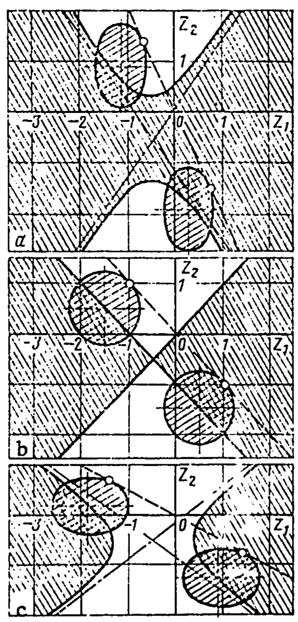

Figure 4.11. Mutual arrangement of the boundaries of the regions of dynamic instability of a space vehicle and the bifurcation boundary

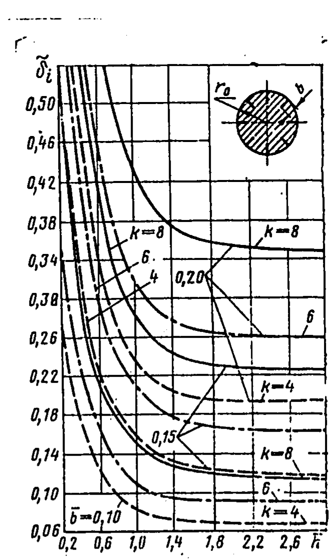

Figure 4.12. Dissipative coefficients as a function of the depth of the liquid for a cylindrical compartment with radial ribs

From Fig 4.11 it follows that on satisfaction of the condition of dynamic instability (Fig 4.25), both the situation $\ell_1>\ell_2$ (in the region denoted by double crosshatching) and the situation $\ell_1<\ell_2$ (not crosshatched) can be realized.

If we take the restriction of smallness of the parameter γ, then depending on the specific compositional layouts of the space vehicle, all forms of the functions $\nu(\eta)$ presented in Fig 4.9 can be realized.

Let us consider the numerical example. Let for the system (4.5)

$$\nu_1=0{,}513;\ \nu_3=0{,}385;\ \nu_5=0{,}656;\ \nu_{1\zeta}=0{,}369;$$
$$\nu_2=0{,}770;\ \nu_4=0{,}150;\ \nu_{2\zeta}=0{,}525,$$
$$\text{then } a_0=0{,}102;\ a_1=0{,}561;\ a_2=0{,}844.$$

In the absence of dissipative forces the investigated system is dynamically unstable. The roots of the characteristic equation have the form $p=\pm0.777\pm5.092i$.

Calculating the characteristic parameters, we find $\ell_1=0.473$; $\ell_2=0.347$; $\theta=0.962$; $g=2.813$.

In the given case $\nu(\eta)$ is a monotonically decreasing function.

The equation of the asymptote has the form

$$\nu=\frac{a_2}{g}\left(\theta+1-\frac{a_1}{a_2}g\right)=0{,}028$$

and indicates that the stabilization of the system is impossible for any values of η if $\nu<0.028$.

In order to insure stability of the system (4.5) in the given case it is necessary to select the value of the parameter $\eta=\varepsilon_2/\varepsilon_1$, for example, $\eta_0=0.5$

From formula (4.31) we calculate the value of ν_* corresponding to the limit of stability of the system (4.5): $\nu_*=\nu(\eta_0)=0.035$.

Selecting the value of $\nu_0\simeq0.04$ corresponding to a 20% stability margin with respect to ν, we find the required pair of stabilizing parameters:

$$\nu_0=0{,}04;\ \eta_0=0{,}5 \text{ or. } \varepsilon_1{}^0=0{,}28,\ \varepsilon_2{}^0=0{,}14.$$

If we assume that the damping coefficients ε_1, ε_2 are identical in both compartments with fuel, the minimum values of ε_1 and ε_2 stabilizing the system (4.5) will be $\varepsilon_1=\varepsilon_2\simeq0.17$.

Let us discuss the problem of practical realization of the damping coefficients found.

Fig 4.12 shows the dissipative coefficients $\tilde{\delta}$ for the cylindrical compartment with radial ribs as a function of the parameters: depth of liquid ($\bar{h}=h/R_0$), number of radial ribs (k) and their relative width ($\bar{b}=b_0/R_0$). Here R_0 is the characteristic dimension of the compartment.

The coefficients $\tilde{\delta}_i$ and ε_i are related by the expression

$$\tilde{\delta}_i=\frac{\pi\varepsilon_i}{\sigma_i \sqrt{\bar{s}_{0i}}} \quad (i=1,2), \qquad (4.36)$$

where σ_i is the characteristic frequency of the fuel oscillations; $\bar{s}_{0i}=s_{0i}/R_0$ is the relative z-coordinates of the free surface of the fuel in the i-th compartment.

Substituting the values of $\varepsilon_1=0.17$; $\sigma_i \approx 5.0$; $\tilde{s}_{0i}=0.09$ in formula (4.36), we find $\tilde{\delta}=0.35$.

Let the depth to which the compartments are filled $\bar{h}_i=h_i/R_0=0.6$. Then as is obvious from the graph in Fig 4.12 in order to insure the required damping coefficients $\varepsilon_1=\varepsilon_2=0.17$ it is necessary, for example, to install either eight radial ribs of relative width $\bar{b}=0.15$ or four radial ribs of relative height $\bar{b}=0.20$. Obviously other versions are possible.

Study of the Joint Effect of the Parameters of the Automatic Stabilization System and Dissipative Forces on the Stability of a Space Vehicle

Let us first consider the equation

$$F_0(x)=x(a_0x^2+a_1x+a_2)+\varrho_0(b_0x^2+b_1x+b_2)+ \varepsilon_1\varepsilon_2(x^2+\varrho_0x)+\varrho_1x[\varepsilon_1(c_0x+1)+\varepsilon_2(d_0x+1)]=0. \qquad (4.37)$$

The structure of this equation is such that the effect of the automatic angular stabilization on the stability of the space vehicle in the oscillation frequency band of the liquid is realized directly (the parameter $\nu=\rho_0$), and also by the parameters $\nu=\varepsilon_1\rho_1$, $\nu=\varepsilon_2\rho_2$. The parameter $\nu=\varepsilon_1\varepsilon_2$ characterizes the effect of the dissipative forces.

The problem consists in selecting these parameters so as to insure (4.20) realness and negativeness of the roots of the equation (4.37). Let us consider each of these parameters separately also using the root hodograph.

a) Effect of the parameter $\nu=\rho_0$.

The equation $F_0=0$, setting $\rho_1=0$, is represented in the form

$$x(a_0x^2+a_1x+a_2)+\nu(b_0x^2+b_1x+b_2)=0. \qquad (4.38)$$

Let us calculate the characteristic (from the point of view of the root hodograph method) points for the equation (4.38):

$x=0;\ x=\frac{1}{2a_0}(-a_1\pm\sqrt{a_1^2-4a_0a_2})$ -- initial points (ν=0); $x=$
$=-1;\ x=-\frac{1}{1-(\nu_4+\nu_5)}$ -- limiting points ($\nu\to\infty$).

The root hodographs for equation (4.38) are presented in Fig 4.13, a.

If for the coefficients a_i, b_i we take the numerical values corresponding to the above-given example, then the values of the parameter ν for which the hodograph intersects the real axis (the stability boundary) are as follows:

$$\nu=-0{,}54;\ \nu=10.$$

These numbers reflect the order of the critical values of the parameter ν for the investigated class of systems.

As is obvious, the best effect is achieved when selecting negative values of the indicated parameter. This requirement is, however, in contradiction to the requirement $\rho_0>0$ following from the conditions of stability of the system in the frequency range of control of the angular movement of the space vehicle as a solid state and in practice cannot be realized.

Thus, the effect of the parameter $\nu=\rho_0$ from the point of view of stability of the space vehicle in the frequency range of the fuel oscillations is unfavorable: an increase in ρ_0 corresponds to an increase in the "degree" of dynamic instability of the system.

b) Effect of the parameter $\nu=\varepsilon_1\varepsilon_2$.

The equation F_0=0 will be written in the form

$$[a_0x^3+(a_1+\varrho_0b_1)x+\varrho_0b_2]+\nu x(x+\varrho_0)=0. \qquad (4.39)$$

The initial points (ν=0) of the root hodograph are the points which for fixed values of the parameter belong to the circle in Fig 4.13, a; the limiting points are the points $x=0;\ x=-\rho_0$.

The trajectories of the roots are presented in Fig 4.13, b, where the dash-dot curve corresponds to displacement of the initial points with an increase in the parameter ρ_0 in the equation (4.38).

As is obvious, the stabilizing effect of introducing the parameter $\nu=\varepsilon_1\varepsilon_2$ is also manifested for $\rho_0\neq0$. Here only the critical value of $\nu=\nu_*$ increases, which for values of $\rho_0\sim1$ typical of the investigated class of objects, can be approximately calculated by the formula

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$$\nu_* = \frac{a_0\alpha^3 + (a_1 + \varrho_0 b_1)\alpha^2 + (a_3 + \varrho_0 b_1)\alpha + \varrho_0 b_2}{\alpha(\alpha + \varrho_0)},$$

where $\alpha=R\ x_1$, x_1 is the complex root of the equation (4.23) for $\nu=0$.

c) Effect of the parameters $\nu=\varepsilon_1\rho_1$; $\nu=\varepsilon_2\rho_1$.

Let us consider the equation

$$(a_0x^2 + a_1x + a_2) + \nu(1 + rx) = 0, \qquad (4.40)$$

where

$$r = \begin{cases} c_0, & \text{if } \nu = \varepsilon_1\varrho_1; \\ d_0, & \text{if } \nu = \varepsilon_2\varrho_1. \end{cases}$$

The root hodograph equation in the plane (μ,Ω) is the equation of a circle

$$\Omega^2 + \left(\mu + \frac{1}{r}\right)^2 = R^2 \qquad (4.41)$$

with the center at the point $(-1/r,\ 0)$ of radius

$$R = \sqrt{\frac{a_2}{a_0} - \frac{a_1}{ra_0} + \frac{1}{r^2}}. \qquad (4.42)$$

The critical value of ν_* (corresponding to the intersection of the μ axis by the circle) is calculated from the expression

$$\nu_* = \frac{1}{2}\left[\left(\frac{2a_0}{r} - a_1\right) + 2\sqrt{a_0(a_0r^{-2} - a_1r^{-1} + a_2)}\right]. \qquad (4.43)$$

It is easy to see that the expressions (4.41)-(4.43) retain their meaning for $a_1^2-4a_0a_2<0$, that is, under the condition of dynamic instability of the space vehicle as an object of control.

In Fig 4.14 we have a family of circles (4.41). When the condition of positive determinacy of the quadratic form corresponding to the kinetic energy of the system represented by the equations (4.5) it follows that $0<r<1$. For $r\to 0$ the centers of the circles (4.41) are shifted in the negative direction of the μ axis. The radius of the circle decreases to a value of $R=\tilde{\omega}$, where $\tilde{\omega}=\text{Im } x$, x is the root of the equation $a_0x^2+a_1x+a_2=0$, and then it increases. Here the length of the least arc subtending the initial points for $0<\nu<\nu_{**}$, and together with it, the critical value of ν_*, increases.

Thus, it is necessary to consider the situation favorable where the center of the circle in Fig 4.14 is located as close as possible to the origin of the coordinates. In this case the value of $\nu_*=\varepsilon_1\rho_1$ is the least.

Let us again consider the preceding example. Calculating the coefficient c_0 and d_0, we find $r=c_0=0.84$ ($\nu_*=0.05$); $r=d_0=0.345$ ($\nu_*=0.63$).

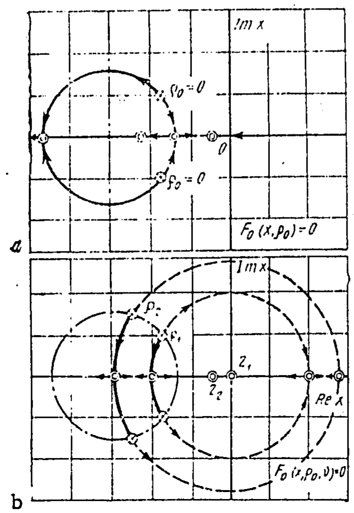

Figure 4.13. Root hodographs of the space vehicle with respect to the parameters ρ_0, $\nu=0$ (case a) and ρ_0, ν (case b)

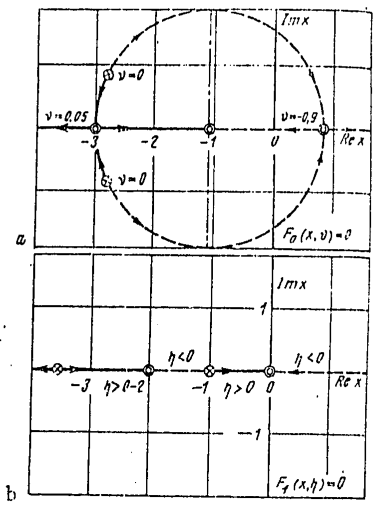

Figure 4.15. Stabilizing parameters ν, η of the space vehicle

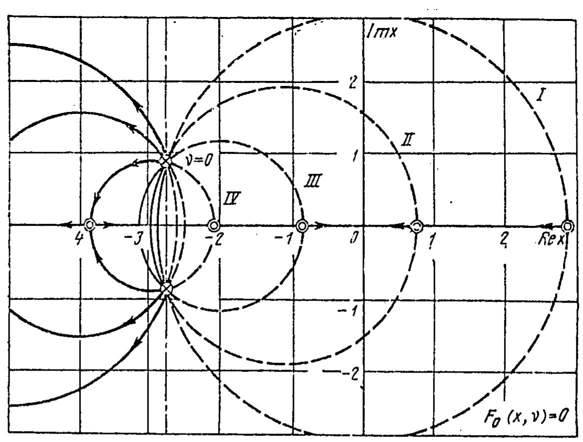

Figure 4.14. Root hodographs of the space vehicle with respect to the parameter ν

As is obvious, the critical value of ν_* essentially depends on the value of the coefficients d_0, c_0 characterizing the connectedness of the partial system (s_1, z), (s_2, z) respectively.

The results obtained above have the following practical significance in the given case.

With invariant adjustment of the parameters of the control system (the values of the parameter ρ_1) it is necessary to insure damping of the oscillations of the partial system characterized by the generalized coordinate s_1. The required value of the damping coefficient ε_2 is minimal in this case (by comparison with the required value of the coefficient ε_1).

Generalizing the equations obtained when investigating the equation (4.38), we note that the condition "b" of the stability criterion (the condition of realness of the roots) can be satisfied by selecting the appropriate values of the parameter

$$\nu=\varepsilon_1\varepsilon_2>0;\ \nu=\varepsilon_1\varrho_1>0;\ \nu=\varepsilon_2\varrho_1>0.$$

From this point of view the indicated parameters are stabilizing parameters, the parameter $\nu=\rho_0$ is destabilizing.

Let us return to the equation

$$F_1(x)=\varrho_1(b_0x^2+b_1x+b_2)+\varepsilon_1x(f_1x+f_2)+\varepsilon_2x(g_1x+g_2)+ +\varepsilon_1\varrho_0x(c_0x+1)+\varepsilon_2\varrho_0x(d_0x+1)+\varepsilon_1\varepsilon_2\varrho_1x=0. \quad (4.44)$$

The structure of equation (4.44) is such that the parameters ρ_1, ε_1, ε_2 influence the magnitude of the roots of this equation directly. The influence of the parameter ρ_0 is indirect (by means of the parameters ε_1, ε_2).

The analysis of the equation (4.44) considering the properties of the system (4.5) indicates that no problems arise connected with insuring reality of the roots of this equation. Therefore primary attention must be given to the problem of selecting the values of the parameters from the given set ε_1, ε_2, ρ_0, ρ_1 such as to insure permutatability of the roots of the equation $F_0=0$; $F_1=0$.

a) Effect of the parameters ε_1, ε_2, ρ_1.

Let us denote $q=\varepsilon_i/\rho_1$ (τ=1,2).

The equation is representable in the form

$$b_0(x+\alpha_1)(x+\alpha_2)+qx(x+\beta)=0; \quad (4.45)$$

$$h_0=f_1+\varrho_0c_0>0;\ h_1=f_2+\varrho_0\ (\ \text{or}\ q=\varepsilon_1/\varrho_1);$$

$$h_0=g_1+\varrho_0d_c>0;\ h_1=g_2+\varrho_0\ (\ \text{or}\ q=\varepsilon_2/\varrho_1).$$

The trajectories of the roots of the equation depend on the mutual arrangement of the characteristic points:

$$x=-\alpha_1;\quad x=-\alpha_2 \text{ (initial points } q=0);$$
$$x=0;\quad x=-\beta \text{ (initial points } q\to\infty).$$

In the case of $\alpha_1<\beta<\alpha_2$ (standard for the investigated class of objects) the root hodographs are segments of the real axis in Fig 4.15, b. The values of the initial and the limiting points in this figure correspond to the numerical example presented above:

$$\beta=2{,}83;\ \alpha_1=1;\ \alpha_2=3{,}55.$$

b) Joint effect of the parameters ε_1, ε_2.

Let us write equation (4.44) in the form

$$\{\varrho_1(b_0x^2+b_1x+b_2)+\varepsilon_1x[(f_1+\varrho_0c_0)x+(f_2+\varrho_0)]\}+ \\ +\varepsilon_2x[(g_1+\varrho_0d_0)x+(g_2+\varrho_0)+\varepsilon_1\varrho_1]=0. \qquad (4.46)$$

In the given case $x=-\alpha_1$, $x=-\alpha_2$ are the roots of the equation (4.46) for $\varepsilon_2=0$. The dependence of α_1, α_2 on the parameter ε_1/ρ_1 was studied above; $x=-\beta$ is the root of the equation (4.46) for $\varepsilon_2\to\infty$. Thus, if we also assume that $\varepsilon_1\rho_1\ll g_2+\rho_0$, the root hodographs for the equation have the form indicated in Fig 4.14. The initial points α_1, α_2 are shifted, but with respect to the points $x=-1$; $x=-[1-(\nu_4+\nu_5)]^{-1}$ as a result of the effect of the parameter ε_1.

On the whole it is possible to see that the parameters ε_1, ε_2 jointly shift the roots of the equation to the right — in the direction of positive μ.

Let us combine the root hodographs corresponding to the equation $F_0=0$; $F_1=0$ (Fig 4.15, a and b) for values of $\nu>\nu_*$ guaranteeing the realness of the roots of the equation $F_0=0$.

From Fig 4.15 it follows that there is a pair of numbers (ν, η) for which coincidence of any two roots of the equations ($F_0=0$; $F_1=0$ and with a further increase in these parameters the condition of permutatability of the conditions, and together with it the condition of stability of the initial system, will be satisfied.

This means that the problem of stabilizing the dynamically unstable space vehicle can be solved by selecting the characteristics of the damping devices with simultaneous adjustment of the parameters of the control system (the automatic stabilization system).

As for the number of values of the corresponding parameters, in each specific case they can be calculated, for example, by the method of successive approximations on the basis of the corresponding hodographs.

Returning to the initial set of parameters

$$\varepsilon_1,\ \varepsilon_2,\ \varrho_0=\frac{k_0 b_{\psi u}}{\sigma^2};\quad \varrho_1=-\frac{k_1 b_{\psi u}}{\sigma}$$

and generalizing the results obtained, it is possible to draw the following conclusions.

1. The dynamic instability of the space vehicle caused by the effect of the following force cannot be eliminated by selecting only the parameters of the automatic stabilization system of selected structure.

2. The damping of the fuel oscillations in the compartments is a defining factor insuring stabilization of a dynamically unstable space vehicle.

3. The requirements on the characteristics of the damping devices can be weakened by special adjustment of the parameters of the automatic stabilization system: namely:

a) Decreasing the dynamic amplification coefficient A (σ) of the stabilization system in the frequency band of the fuel oscillations;

b) Increasing the lead ϕ (σ) of the automatic stabilization system on these frequencies.

The limiting values of the corresponding parameters can be calculated from equation (4.31).

4.4. Stabilization of Structurally Unstable Space Vehicles Using a Discrete Stabilization Algorithm

Preliminary Remarks

The method of reducing the dynamic amplification coefficient A (σ) in the required frequency range discussed in this section is based on the idea of separating the movement of the space vehicle into slow and fast components (in the form in which it was proposed by I. M. Sidorov [68]).

In order to realize the corresponding algorithm, the presence of an on-board computer (an on-board digital computer) is required.

For the following discussion the problems that are typical of the discrete automatic control systems will not be considered. Primary attention will be given to estimating the possibilities of the algorithm in the sense of attenuating the effect of the structural instability of the objects with liquid-propellant rocket engines.

Let us consider the following dynamic system [85]:

$$A\ddot{\vec{x}}+B\dot{\vec{x}}+C\vec{x}+\vec{D}u=\vec{F}(t),\ \vec{x}(0)=\vec{x}_0,$$

where $\vec{x}$, $\vec{D}$, $\vec{F}$, n are the vectors; A, B, C are the matrices of dimensionality nXn; u(t) is a scalar function subject to definition.

The variables $\vec{x}_i$ (i=1,2,...,n) will be separated into two groups:

1) $\vec{x}_1$, $\vec{x}_2$,...,x_k (k<n) -- "slow" variables accessible to measurement;

]) $\vec{x}_{k+1}$,...,$\vec{x}_{n-1}$, $\vec{x}_n$ -- "fast" variables.

Let us assume that the control input u(t) is formed using slow variables.

Together with the system (C) describing the controlled movement of the real object, let us consider another system representing the model of the object:

$$a\ddot{\vec{y}}+b\dot{\vec{y}}+c\vec{y}=\vec{d}u+\vec{f}(t), \qquad (M)$$

where $\vec{y}$, $\vec{d}$, $\vec{f}$ are m-vectors; a, b, c are the matrices of dimensionality mXm.

In real situations m<<n, for the model (M) reflects our knowledge of the object which, as a rule, is far from sufficient to fully reproduce the system (C).

It is expedient to select the model beginning with the physical essence of the latter based, for example, on the experimental data of investigation of the dynamic properties of real systems. This problem -- the identification problem -- is of independent interest, and it will not be considered in this paper.

Let us propose for determinacy that the order of the model (the dimensionality of the vector $\vec{y}$) corresponds to the number of slow variables which are, therefore, controllable variables.

Thus, we have two systems (C)-(M). The problem consists in constructing the algorithm which will insure stability and the required quality of control of the system (C) for given disturbances $\vec{F}(t)$, and at the same time will weaken the effect on the transient processes of the oscillations of auxiliary oscillators (the effect of "structural instability").

Let us then propose that the parameters of the model (the matrices, a, b, c and the vector d) be selected and in the process of functioning of the algorithm not change. Then in accordance with the discussed requirements the proposed algorithm must solve the following problems:

Processing the information about the movement of the system (C) (according to the sensor readings) and the movement of the model (by the results of integrating the equations (M));

Calculation of the initial conditions and prediction of the disturbances for the model (by the results of processing the information);

Calculation (prediction) of the controlled inputs u(t) for the system (C) for a finite time interval in advance.

As is obvious, in the given case the adaptation is understood in the sense that by adjustment of the disturbance vector f and the vector of the initial conditions $\vec{y}(0)$ the model will strive to describe in the best way the slow component of the movement of the system under the conditions of variable external disturbances $\vec{F}(t)$ and the presence of interference in the form of the effect of fast components of the solution.

Stabilization Algorithm

As applied to the objects of the investigated class the stabilization algorithm can be executed, for example, in the following form.

Without restricting the generality, let us propose that the equations of motion of the space vehicle in the active segment, in addition to the generalized coordinates z, ψ characterizing its movement as a solid body (slow variables) also include the generalized coordinates s_1, s_2 caused by movement of the fuel in the tanks for the combustible fuel component and the oxidizing agent (fast variable).

We have the following equations.

The system:

$$A\ddot{\vec{x}}+B\dot{\vec{x}}+C\vec{x}+\vec{D}u=\vec{F}(t);\ \vec{x}(0)=\vec{x}_0; \qquad (C)$$

$$\vec{x}=\begin{pmatrix} z \\ \psi \\ s_1 \\ s_2 \end{pmatrix};\ A=\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -a_{\psi s_1} & -a_{\psi s_2} \\ -a_{s_1 z} & -a_{s_1\psi} & 1 & 0 \\ -a_{s_2 z} & -a_{s_2\psi} & 0 & 1 \end{pmatrix};$$

$$B=\begin{pmatrix} 0 & -a'_{z\psi} & 0 & 0 \\ 0 & -a'_{\psi\psi} & 0 & 0 \\ 0 & 0 & -\varepsilon_1 & 0 \\ 0 & 0 & 0 & -\varepsilon_2 \end{pmatrix};\ C=\begin{pmatrix} 0 & -a_{z\psi} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma_1^2 & 0 \\ 0 & 0 & 0 & -\sigma_2^2 \end{pmatrix};\ D=\begin{pmatrix} a_{zu} \\ a_{\psi u} \\ 0 \\ 0 \end{pmatrix}.$$

The model

$$a\ddot{\vec{y}}+b\dot{\vec{y}}+c\vec{y}+du=\vec{f}_\tau(t);\ \ \vec{y}_\tau(0)=\vec{y}_{\tau 0};\ (M_\tau);$$

$$\vec{y}=\begin{pmatrix} \zeta \\ \theta \\ 0 \end{pmatrix};\ a=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$b=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{k_3}{a_0} & \frac{k_1}{a_0} & -\frac{a_1}{a_0} \end{pmatrix};\ c=\begin{pmatrix} 0 & a_{\zeta\theta} & a_{\zeta u} \\ 0 & a_{\theta\theta} & a_{\theta u} \\ \frac{k_2}{a_0} & \frac{k_0}{a_0} & -\frac{1}{a_0} \end{pmatrix};\ \vec{f}=\begin{pmatrix} f_1 \\ f_2 \\ 0 \end{pmatrix}.$$

As is obvious, the system of equations of the model also includes the equation of the control system which is taken in the form

$$a_0\ddot{u}+a_1\dot{u}+u=k_0\theta+k_1\dot{\theta}+k_2\zeta+k_3\dot{\zeta}.$$

Simultaneously with the system (M) let us also consider a "fictitious" system

$$a\ddot{\vec{y}}+b\dot{\vec{y}}+c\vec{y}+\vec{d}u=\vec{f}_T(t);\ \vec{y}_T(0)=\vec{y}_{T0},$$

differing from (M) only by the vectors (y_T) of the initial conditions and disturbances. Thus, there are two pairs of vectors

$$\vec{y}_\tau(0),\ \vec{f}_\tau(0)\ \text{и}\ [\vec{y}_T(0);\ \vec{f}_T(0)].$$

Let us introduce three characteristic time intervals:

h is the integration step of the systems (M_τ), (M_T) and the system (C) during simulation (it coincides with discreteness of arrival of the information);

τ is the time interval between two successive adjustments of the model (M_τ): the vectors $\vec{y}_\tau$ (0) and $\vec{f}_\tau$ respectively (inside the interval $[k_\tau,\ (k+1)_\tau]$ the vector f_τ=const).

T is the length of the initial data processing interval -- it coincides with the time interval of adjustment of the vectors $y_T(0)$, $f_T(0)$ in the system (M_T).

Let us also introduce the notation: s -- the number of adjustments of the vectors $\vec{y}_\tau$, $\vec{f}_\tau$ in the analysis section; N -- the number of integration steps in the segment of length T.

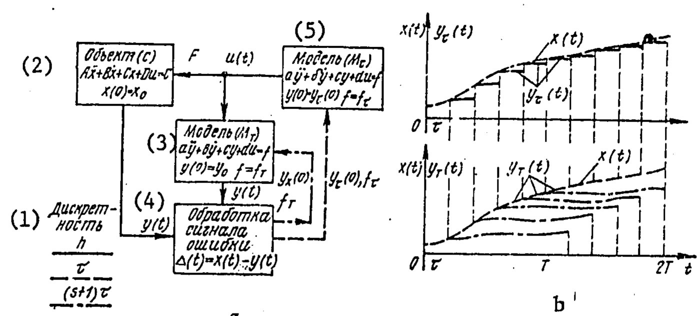

Figure 4.16. Stabilization algorithm of a space vehicle: a -- schematic diagram; b -- regime τ, T adjustment of the model of the space vehicle

Key:

1 -- discreteness; 2 -- object; 3 -- model; 4 -- error signal processing; 5 -- model

Then the following expressions exist between the variables h, τ, T:

$$T=(s+1)\tau=Nh.$$

Let us make some remarks with respect to the role of each of the systems (C), (M_τ) and (M_T) in the algorithm.

The system (C) is a dynamic system which is the object of control; the coordinates $x_1(t)=z(t)$; $x_2(t)=\psi(t)$, information about which comes to the computer with a step size of h, are assumed to be accessible to measurement.

The system (M_τ) is a model of the system (C); it takes into account only part of the generalized coordinates (the "slow" variables) and it is designed for processing the control input u(t), coming with a step size h to the input of the system (C). The initial conditions $\vec{y}_\tau$ (0) and the disturbance vector f_τ are adjusted every τ seconds on the basis of the results of the information processing.

The system (M_T) is designed for calculation of some fictitious movement of the model in the interval T (Fig 4.16, b), without considering its adjustment inside the interval in order to compare with real movement and error formation:

$$\Delta_T^{(i)}(t)=x_T^{(i)}(t)-x^{(i)}(t)\ (i=1,\ 2). \qquad (4.47)$$

In order to process the mismatch signal, the least squares method is used which separates the low-frequency component from the complex signal $\Delta_T(t)$, which is used in the control law u=u(t).

The block diagram of the algorithm is presented in Fig 4.16, a. The representation of the operation of the algorithm in the segment (0,2T) gives Fig 4.16, b where the dotted line indicates the actual movement of the system (C), and the solid line, the movement of the model (M_τ); the dot-dash line indicates movement of the model (M_T) (Fig 4.16, b). The breaks at the points h_τ (Fig 4.16, b) and $(s+1)\tau$ denote adjustment of the initial conditions for the systems (M) and (C) at the corresponding points in time.

Let us discuss some significant events in the operation of the algorithm. The algorithm has a finite memory (the length of the memory interval is T) in which the following are stored:

Information about the movement of the system (C) in the form of the matrix MC of capacity nXN which is denoted by means of a shift of the matrix elements through the time intervals τ as the system moves;

Information about the movement of the fictitious system in the form of the matrix MΦ of capacity mXN renewed as follows;

Intermediate results, including the results of calculating the proposed external forces (prediction of the interval of the controlling forces ahead, the prediction ahead of the initial conditions for the systems (C) and (M_τ) for the interval τ.

When simulating a real object by the system (C) it is also necessary to integrate the system (C) for which it is necessary to provide the corresponding memory modules.

It is necessary to add a number of constants to this (the length of the analysis segments, the switching segment, the number of switches, the constant for the least squares method, and so on).

Let us consider a characteristic point in time t=(k+1)T. Let us discuss the following steps:

a) Processing of information (filtration of the periodic components in the control signal)

Using the data accumulated in the MC, MΦ modules, let us calculate

$$\Delta_1(t)=Z(t)-\zeta(t);\ \ \Delta_2(t)=\psi(t)-\theta(t)$$

at the time

$$t_1=(kT+h),\ldots,\ t_N=(kT+Nh)=(k+1)T.$$

The analyzed functions $\Delta_i(t)$ (i=1,2) are the sum of the low-frequency process caused by the presence of external disturbances and the sum of the finite number (in the given case two) of harmonics connected with the presence of oscillatory elements with frequencies.

With respect to the essence of the problem it is necessary to insure filtration of the periodic components of the signal Δ_i.

Let us set

$$\delta_1(t)=\alpha_2 t^2+\alpha_1 t+\alpha_0;\ \ \delta_2(t)=\beta_2 t^2+\beta_1 t+\beta_0. \qquad (4.48)$$

In accordance with the least squares method, in order to find the vectors $\vec{\alpha}=(\alpha_2,\ \alpha_1,\ \alpha_0)$ $\vec{\beta}=(\beta_2,\ \beta_1,\ \beta_0)$ let us write the system

$$\mu\vec{\alpha}=\vec{b}_1;\ \ \mu\vec{\beta}=\vec{b}_2, \qquad (4.49)$$

where $\vec{b}_1=(b_{12},\ b_{11},\ b_{10}),\ \vec{b}_2=(b_{22},\ b_{21},\ b_{20});$

$$\mu=\begin{pmatrix}\mu_4 & \mu_3 & \mu_2\\ \mu_3 & \mu_2 & \mu_1\\ \mu_2 & \mu_1 & \mu_0\end{pmatrix};$$

The elements of the vectors $\vec{b}_1$, $\vec{b}_2$ and the matrix μ are calculated by the formula

$$\mu_0=N;\ \mu_1=\frac{N(N+1)}{2}\left(\frac{T}{N}\right);\ \mu_2=\frac{N(N+1)(2N+1)}{6}\left(\frac{T}{N}\right)^2;$$

$$\mu_3=\left[\frac{N(N+1)}{2}\right]^2\left(\frac{T}{N}\right)^3;\ \mu_4=\frac{N(N+1)(2N+1)(3N^2+3N-1)}{30}\left(\frac{T}{N}\right)^4;$$

$$b_{j2}=\left(\frac{T}{N}\right)^2\sum_{i=1}^{N} i^2\Delta_{ji};\ b_{j1}=\left(\frac{T}{N}\right)\sum_{i=1}^{N}\Delta_{ji};\ \ b_{j0}=\sum_{i=1}^{N}\Delta_{ji}.$$

From the formulas (4.49) we find

$$\vec{\alpha}=\mu^{-1}\vec{b}_1;\ \vec{\beta}=\mu^{-1}\vec{b}_2. \qquad (4.50)$$

b) Calculation of the disturbances (prdicting ahead by the interval τ)

Let us propose that the difference in the values of x(t), y(t) is caused by the effect of various (constant) disturbances (F, M) and (f_{1T}, mT) in the systems (C) and (M_T).

Then $\ddot{\Delta}_1(t)=\Delta F;\ \ddot{\Delta}_2(t)=\Delta M,$

where

$$\Delta F=F-f_T;\ \Delta M=M-m_T \qquad (4.51)$$

are the proposed values of the disturbances at the time t=kT for the system (M_T).

Comparing expressions (4.51) with (4.48), we find

$$f_1[(k+1)T]=f_1(kT)+2\alpha_1;\ f_2[(k+1)T]=f_2(kT)+2\beta_2. \quad (4.52)$$

c) Calculation of the initial conditions for the systems (M_τ and M_T).

The values of

$$\delta_1[(k+1)T]=\alpha_2T^2+\alpha_1T+\alpha_0;\ \ \dot{\delta}_1[(k+1)T]=2\alpha_2T+\alpha_1;$$
$$\delta_2[(k+1)T]=\beta_2T^2+\beta_1T+\beta_0;\ \dot{\delta}_2[(k+1)T]=2\beta_2T+\beta_1$$

give the difference (on the average) of the values of the functions z(t), ζ(t) and ψ(t), θ(t) and their derivatives at the time t=(k+1)T.

Thus, if we set

$$\zeta(0)=\zeta[(k+1)T]+\alpha_2T^2+\alpha_1T+\alpha_0;\ \dot{\zeta}(0)=\dot{\zeta}[(k+1)T]+2\alpha_2T+\alpha_1;$$
$$\theta(0)=\theta[(k+1)T]+\beta_2T^2+\beta_1T+\beta_0;\ \dot{\theta}(0)=\dot{\theta}[(k+1)T]+2\beta_2T+\beta_1; \quad (4.53)$$
$$u(0)=u[(k+1)T]\ \ \dot{u}(0)=\dot{u}[(k+1)T],$$

and also $f_1=f_1(kT)+2\alpha_2;\ f_2=f_2(kT)+2\beta_2,$

then the solution of the system of equations (M_τ) on the basis of the known principles of the theory of differential equations will on the average be closed to the solution of the system (C) in the next (sufficiently small) time interval τ.

As for the system (M_T), the calculated values of (ζ_0, θ_0, u_0) and (f_1, f_2) will serve as the initial conditions for calculating a fictitious movement in the segment

$$\{(k+1)T,\ [(s+1)\tau+(k+1)T]\}.$$

Thus, all of the required initial data are calculated for integration of the systems (M_τ) and (M_T) in the interval

$$[(k+1)T;\ (k+1)T+\tau].$$

The further sequence of the calculations is as follows:

The system (M_τ) is integrated in an interval of length h;

The system (C) is integrated for the calculated values of u(t) as a result of integration of the system (M_τ);

The system (M_T) is integrated;

The results of the calculations are stored in the digital computer memory.

Finally, at the time $t=(k+1)T+\tau$, the entire "a" and "b" cycle is repeated.

The effect of stabilization of the space vehicle using the discussed algorithm consists in the following. As is demonstrated in reference [23] the following expression exists relating the real part $\mathrm{Re}\,p_j$ of the root p_j of the characteristic equation of the closed system made up of the object and the control system:

$$\mathrm{Re}\,p_j=-\frac{\varepsilon_j}{2}+\frac{A_u(\Omega_j)\sin\varphi_u(\Omega_j)}{2}\,\frac{\Phi_k(\omega_j)}{\Phi_0'(\omega_j)};\qquad(4.54)$$

$$\Omega_j=\omega_j T,\ A_u(\Omega_j)\sim\frac{1}{\Omega_j},\qquad(4.55)$$

where ω_j is the frequency characterizing the investigated control system; T is the characteristic information analysis time during functioning of the algorithm.

In the case of structural instability of the object the second term in formula (4.54) is positive. Increasing the parameter T, on the basis of the expression (4.55) we decrease the dynamic amplification coefficient of the control system $A_u(\Omega_j)$.

For some T_*, $\mathrm{Re}\,p_j$ becomes positive, and therefore the effect of the structural instability of the space vehicle for the object of control will be eliminated.

Consequently, the main thing here is the possibility of lowering the dynamic amplification coefficient $A_u(\Omega_j)$ in the selected frequency range ω_j of the object, which usually is difficult to realize for a control system of classical structure [38].

Numerical Realization of the Algorithm*

The following program is a numerical realization of the adaptive control method for the case of two auxiliary oscillators (8-th order system), and it is written in FORTRAN IV. The algorithm was checked out and the numerical calculations were performed on the SIEMENS-4004/45 computer.

The operation of the algorithm begins with input of the initial information containing the following:

The coefficients of the equations describing the system as an object of control;

The coefficients of the equations describing the "model" of the system;

The time intervals n, τ, T.

The module for calculating the matrix of the coefficients μ for the least squares method operates in the initial program, for these values depend only on the number of points N in the information analysis section T.

The equation conversion module reduces all of the systems of equations to the form convenient for application of the standard program for integrating systems of differential equations by the Runge-Kutta method. By the "integration" results the matrices of MC are formulated for the system, and the matrices MF1, MP2, MF3, MP4 for the "models."

The matrix MY is filled with the values obtained for the control inputs. The matrix Δ is the table of mismatches between the actual behavior of the system and the operating model.

The variables h_ℓ, τ_c are used to organize the cycles inside the program; the variable s makes it possible to increase the information analysis interval.

---

*The algorithm was programmed and checked out on the computer by O. B. Makhlin and T. K. Chudakova.

Description of Files and Variables

```
1     PROGRAM DUSHA
2     REAL DMU(3,3)/9*0.0/, DMUOBR(3,3)/9*0.0/, AC(4,4), BC(4,4), CC(4,4,
3     IDC(4), FC(4), ACOBR(4,4)/16*0.0/

4     REAL G1(4,4), G2(4,4), G3(4), G4(4), APRC(8,8)/64*0.0/,
      BPRC(8)/8*0.0/
5     1, FPRC(8)/8*0.0/
6     REAL AM(3,3), BM(3,3), CM(3,3), MM(3), ALFA(6,6)/36*0.0/,
      FM(6)/6*0.0/
7     REAL BINK(3), B2NK(3), ALFANK(3), BETANK(3)
8     REAL DELTA(2,25)/50*0.0/
9     REAL *8 APRCD(8,8), ALFAD(6,6), YOMD(15)/15*0.1/,YOCD(15),
10    /YKMD(15), YKCD(15),
11    +XAD, XED, A(16,25),
12    +FMD(6), BPRCD(8),FPRCD(8),USYS,USYSO,USYSI,HD
13    REAL MB(2,5), MC(8,25)/200 *0.001/, MU(2,25)/50*0.0/,
14    +MHY(5,6)/30* 0.0/
15    REAL MF1(%,25)/125 *0.001/,MP2(5,25)/125*0.001/,
16    +MF3(5,25)/125*0.001/,MP4(5,25)/125*0.001/
17    REAL MMPU(4,25)/100*0.001/
```

Assignment of Time Intervals

```
18    EXTERNAL WHODM
19    EXTERNAL WHODF
20    EXTERNAL WHODC
21    RO-6.0
22    H=0.05
23    S=4.0
24    TAU=0.25
25    N=25
```

Calculation of the Matrix μ

```
26 C *** BLOCK 2*** Calculation of the coefficient μ for the least squares
      method,
27 C  Obtaining the matrix μ and manipulation of it
28    IC4S=0
29    T=(S+1.0)* TAU
30    DN=N
31    UMO=DN
32    UM1=N*(N+1.0)*H/2.0
33    UM2=N*(N+1.0)*(2.0*N+1.0)*(H**2)/6.0
34    UM31=(N*(N+1.0)/2.0)
35    UM3=(UM31**2)*(H**3)
```

```
      UM41=N*(N+1.0)*(2.0*N+1.0)
      UM42=3.0*(N**2)+3.0*N=1.0
      UM4=UM41*UM42*(H**4)/30.0
      DMU(1,1)=UM4
      DMU(2,1)=UM3
      DMU(3,1)=UM2
      DMU(1,2)=UM3
      DMU(2,2)=UM2
      DMU(3,2)=UM1
      DMU(1,3)=UM2
      DMU(2,3)=UM1
      DMU(3,3)=UM0
      WRITE(99,30) ((DMU(1,K),K=1,3),I=1,3)
   30 FORMAT ('',10X, 'MATRIX MU'//('',3(3X,F11.6)))
      DO 50 I=1,3
```

Manipulation of Matrix μ

```
      DO 50 Ch 1,3
   50 DMUOBR(I,K)=DMU(I,K)
      CALL SPINV(DMUOBR, 3,3,ISIG)
      WRITE(99,60) ((DMUOBR(I,K),K=1,3), I=1,3)
   60 FORMAT('',10X,'MATRIX MU MANIPULATE'//('',3(3X,F11.67)))
```

Input of System Coefficients

```
C*** BLOCK 3*** CONVERSION OF SYSTEMS O
      READ(97,2) ((AC(I,K),K=1,4),I+1,4)
      READ(97,2) ((BC(I,K),K=1,4)I=1,4)
      RESD(97,2) ((CC(I,K),K=1,4) I=1,4)
    2 FORMAT(4F9.5)
      READ(97,2) (DC(I),I=1,4)
      READ(97,2)(FC(I),I=1,4)
```

Conversion of System Equations

```
      DO 3 I=1,4
      DO 3 K=1,4
    3 ACOBR(I,K)=AC(I,K)
      CALL SPINV(ACOBR,4,4,ISIG)
      CALL MAMURA(ACOBR,FC,G4,4,4,1)
      CALL MAMURA(ACOBR,DC,G3,4,4,1)
      CALL MAMURA(ACOBR,CC,G],4,4,4)
      CALL MAMURA(ACOBR,BC,G1,4,4,4)
      DO 6 I=1,8
      DO 6 K=1,8
    6 APRC(I,K)=0.0
      DO 8 K=1,4
```

```
8       APRC(2*K=1,2*K)=1,0
        DO 10 I=1,4
        DO 10 K=1,4
10      APRC(2*1,2*K=1)=G2(I,K)
        DO 12 I=1,4
        DO 12 K=I,4
12      APRC(2*1,2*K)=G1(I,K)
        DO 14 I=1,4
        BPRC(2*I)=G3(I)
14      BPRC(2*I=1)=0.0
        DO 15 I=1,4
        FPRC(2*I)=G4(I)
15      FPRC(2*I=1)=0.0
        WRITE(99,16) ((APRC(I,K),K=1,8), I=1,8)
16      FORMAT('',10X, 'MATRIX A SYSTEM CONVERSION'//('',8(IX
        F11.6)))
        WRITE99,17) (BPRC(I), I=1,8)
17      FORMAT('',10X, 'MATRIX B SYSTEM CONVERSION'//'',8(IX,F11.6))
        WRITE(99,18)(FPRC(I),I=1,8)
18      FORMAT('',10X,'MATRIX F SYSTEM CONVERSION'//'',8(IX,F11.6))
```

Model Coefficient Input

```
C***    BLOCK 4***MODEL CONVERSION
        READ(97,100((AM(I,K),K=1,3),I=1,3)
100     FORMAT(9F7.3)
        READ(97,100)((BM(I,K),K=1,3),I=1,3)
        READ(97,100)((CM(I,K),K=1,3),I=1,3)
        READ(97,100)(MM(I),I=1,3)
110     FORMAT(3F7.3)
```

Conversion of the Model Equations

```
        DO 330 I=1,2
        DO 330 K=1,5
330     MB(I,K)=MM(I)
        DO 120 I=1.6
        DO 120 K=1,6
120     ALFA(I,K)=0.0
        DO 130 K=1,3
130     ALFA(2*K=1,2*K)=1.0
        DO 140 I=1,3
        DO 140 K=1,3
140     ALFA(2*1,2*K=1)=CM(I,K)
        DO 150 K=1,3
        DO 150 I=1,3
150     ALFA(2*1,2*K)=BM(I,K)
        DO 160 I=1,6
160     FM(I)=0.0
```

```
      DO 170 I=1,3
      FM(2*I)=MM(I)
170   FM(2*I=1)=0.0
      WRITE(99,180) (ALFA(I,K),K=1,6),I-1,6)
180   FORMAT('',10X,'MATRIX ALFA OF MODEL'//('',6(3X,F11.6)))
      WRITE(99,200)(FM(I),I=1,6)
200   FORMAT('',10X,'VECTOR F OF MODEL'//'',6(3X,F11,6)
      DO 202 I=1,8
      DO 202 K=1,8
      APRCD(1,K)=DBLE(APRC(I,K))
      BPRCD(K)=DBLE(BPRC(K))
202   FPRCD(K)=DBLE(FPRC(K))
      DO 203 I=1,6
      DO 203 K=1,6
203   ALFAD(I,K)=DBLE(ALFA(I,K))
```

Organization of τ Loop

```
C***  BLOCK 6
      TAUC4=TAU
```

Organization of h Loop

```
C***  BLOCK 7
1007  HC4=0.0
      HC4=H
      NSTB=S*TAU/H+0.5
```

Integration of "Control" Model

```
C***  BLOCKS 8,9 INTEGRATION OF MODEL S+1 TIMES
1008  XAD=HC4-H
      XED=HC4
      IS1=S+1,5
      FM(2)=MB(1,IS1)
      FM(4)=MB(2,IS1)
      DO 410 I=1,6
410   FMD(I)=DBLE(FM(I))
      DO 420 I=1,4
420   YOMD(I)=DBLE(MMPU(I,NSTB))
      YOMD(5)=DBLE(MU(1,NSTB))
      YOMD(6)=DBLE(MU(2,NSTB))
      WRITE(99,44) TAUC4,XAD,XED,FM(2),FM(4)
440   FORMAT(//'','*******INTEGRATION OF CONTROL MODEL
      FOR TAU='
      +,F5.1,2X,'H=',F5,1,'-',F5,1/10X,'FM(2)=',F11.6,3X,'FM(4)=',
      +F11.6)
      CALL PR4MOD(ALFAD,FMD)
      CALL DRKU(6,WHODM,XAD,YOMD,XED,YKMD,
      +5,0.01,2,A)
```

```
      DO 430 I=1,4
430   MMPU(I,NSTB+1)=SNGL(YKMD(I))
      MU(1,NSTB+1)=SNGL(YKMD(5))
      MU(2,NSTB=1)=SNGL(YKMD(6))
```

Integration of "Fictitious" Model

```
      WRITE(99,300) TAUC4, XAD,XED
300   FORMAT('',10X,'INTEGRATION OF MODEL FOR TAU=',
      F5,1,2X,'H=',
      +F5,1,'-',F5.1)
      DO 240 JJ=1,IS1
      FM(2)=MB(1,JJ)
      FM(4)=MB(2,JJ)
      DO 230 I=1,6
230   FMD(I)=DBLE(FM(I))
      ICS1=IS1-JJ+1,1
      WRITE(99,310)ICS1
310   FORMAT('',20X,'INTEGRATION',14,'ROWS(NUMBERS FROM THE TOP)')
      WRITE(99,450)FM(2),FM(4)
450   FORMAT('',10X,'FM(2)=',F11.6,3X,'FM(4)=',F11.6)
      YOMD(1)=DBLE(MF1(ICS1,NSTB))
      YOMD(2)=DBLE(MP2(ICS1,NSTB))
      YOMD(3)=DBLE(MF3(ICS1,NSTB))
      YOMD(4)=DBLE(MP4(ICS1,NSTB))
      USYS0=DBLE(MU(1,NSTB))
      USYS1=DBLE(MU(1,NSTB+1))
      HD=DBLE(H)
      CALL FMOD(ALFAD,FMD,USYS0,USYS1,XAD,HD)
      CALL DRKU(4,WNODF,XAD,YOMD,XED,YKMD
      +5,0.01,2,A)
      MF1(ICS1,NSTB+1)+SNGL(YKMD(1))
      MP2(ICS1,NSTB+1)=SNGL(YKMD(2))
      MF3(ICS1,NSTB+1)=SNGL(YKMD(3))
      MP4(ICS1,NSTB+1)=SNGL(YKMD(4))
```

System Integration

```
240   CONTINUE
C***  BLOCKS 10,11 INTEGRATION OF SYSTEM
      USYS0=DBLE(MU91,NSTB))
      USYS1=DBLE(MU(1,NSTB+1))
      HD=DBLE(H)
      WRITE(99,320) TAUC4,XAD,XED
320   FORMAT('',10X,'INTEGRATION OF SYSTEM FOR',10X,'TAU=
      ',F5.1,
      +'XA=',F5.1,'XE=',F5.1)
      DO 250 I=1,8
250   YOCD(I)=DBLE(MC(I,NSTB))
```

```
      CALL PR4SYS(APRCD,BPRCD,FPRCD,USYSO,USYS1,XAD,HD)
      CALL DRKU(8,WNODC,XAD,YOCD,XED,YKCD,
      +5,0.01,2,A)
      DO 260 I=1.8
260   MC(I,NSTB+1)=SNGL(YKCD(I))
C***  BLOCK 12
      IF(HC$.LT.TAU) GO TO 10013
      GO TO 10014
10013 HC4=HC4+H
      NSTB=NSTB+1
      GO TO 1008
```

Formation of Matrix Δ

```
C***  BLOCK 14***FORMATION OF MATRIX DELTA
      IK=S+1.5
      DO 500 K=1,N
      DELTA(1,K)=MC(1,K)=MF1(IK,K)
      DELTA(2,K)=MC(3,K)=MF3(IK,K)
500   CONTINUE
```

Calculation of the Righthand Sides of the System in the Least Squares Method

```
C***  BLOCK 15***CALCULATION OF RIGHTHAND SIDES FOR THE LEAST
      SQUARE METHOD
      DO 510 I=1,3
      BINK(i)=0.0
510   B2NIK(I)=0.0
      DO 520 K=1,N
      BINK(1)=BINK(1)+(H**2)*(K**2)*DELTA(1,K)
      BINK(2)=BINK(2)+(H*K*DELTA(1,K)
      BINK(3)=BINK(3)+DELTA(1,K)
      B2NK(1)=B2NK(1)+(H**2)*(K**2)*DELTA(2,K)
      B2NK(2)=B2NK(2)+(H*K*DELTA(2,K))
520   B2NK(3)=B2NK(3)+DELTA(2,K)
      WRITE(99,530)((DELTA(I,K),K=1,10),I=1,2)
      WRITE(99,530)((DELTA(I,K),K=11,20),I=1,2)
      WRITE(99,531)((DELTA(I,K),K=21,25),I=1,2)
531   FORMAT('',10X,'DELTA MATRIX FOR THE LEAST SQUARES METHOD'/
      ('',5
      +(1X,F11,6)))
530   FORMAT('',10X,'MATRIX DELTA FOR THE LEAST SQUARES METHOD'
      /('',10
      +(1X,F11,6)))
      WRITE(99,540) (BINK(I),I=1,3)
      WRITE(99,550) (B2NK(I),I=1,3)
540   FORMAT('',10X,'LEFTHAND SIDE OF THE LEAST SQUARES METHOD','B1'
      /('',3(5X,F11
      1.6)))
550   FORMAT('',10X,'LEFTHAND SIDE...B2'/'',3(5X,F11.6)
```

```
C***  BLOCK 16***OBTAINING THE COEFFICIENTS OF PARABOLAS
      CALL MAMURA(DMUOBR,BINK,ALFANK,3,3,1)
      CALL MAMURA(DMUOBR,B2NK,BETANK,3,3,1)
```

Calculation of the Coefficients of the Parabolas $\delta_1(t)$, $\delta_2(t)$

```
      WRITE(99,560)(ALFANK(I),I=1,3)
      WRITE(99,570)(BETANK(I),I=1,3)
560   FORMAT('',10X,'COEFFICIENT OF PARABOLA ALPHA'/'',(5X,F11,6))
570   FORMAT('',10X,'COEFFICIENT OF PARABOLA BETA'/'',(5X,F11,6))
```

Calculation of the Disturbances f_1, f_2 for the Model

```
C***  BLOCK 17***CALCULATION OF DISTURBANCE FOR THE MODEL F1
      FM(2)=MB(1,1)+2.0*ALFANK(1)
      FM(4)=MB(2,1)+2.0*BETANK(1)
      DO 590 I=1,2
      DO 591 K-1,4
591   MB(I,K)=MB(I,K+1)
590   MB(1,5)=FM(2*I)
      MB(1,5)=FM(2)
      MB(2,5)=FM(4)
      WRITE(99,580)(FM(I),I=1,6)
```

Calculation of New Initial Conditions of the Model

```
580   FORMAT('',10X,'NEW VECTOR F OF THE MODEL '/'',6(5X,F11,6))
C***  BLOCK 18***CALCULATION OF NEW INITIAL CONDITIONS OF THE MODEL
      STH=S*TAU/H+0.5
      SITH=(S+1)*TAU/H+0.5
      KSTH=STH
      KSITH=SITH
      KS1=S+1,5
      MF1(I,KSTH)=MF1(KS1,KS1TH)+ALFANK(1)*T**2+
      +ALFANK(2)*T+ALFANK(3)
      MP2(1,KSTH)=MP2(KS1,KS1TH)+2.0*ALFANK(1)*T+
      +ALFANK(2)
      MF3(1,KSTH)=MF3(KS1,KS1TH)+BETANK(1)*T**2+
      +BETANK(2)*T+BETANK(3)
      MP4(1,KSTH)=MP4(KS1,KS1TH)+2.0*BETANK(1)*T+
      +BETANK(2)
      WRITE(99,600)MF1(1,KSTH),MP2(1,KSTH),MF3(1,KSTH),
      MP4(1,KSTH)
600   FORMAT('',10X,'NEW INITIAL CONDITIONS OF THE MODEL'//'
      ','Y1(S+1)=',F1
      11.6,2X,'Y2(S+1)=',F11.6,2X,'Y3(S+1)=',F11.6,2X,
      'Y4(S+1)=',F11.6)
      IC4S=IC4S+1
      II=S+1.1
      IF(IC4S-II)20000,20001,20000
```

```
 20001   WRITE(99,20002)TAUC4,IC4S
 20002   FORMAT(//'',10X,'TAU=',F5.1,5X,'IC4S=',12)
         IC4S=0
         WRITE((9,610)TAUC4
  610    FORMAT(///'',10X,'PRINTOUT OF THE MATRICES AFTER TAU=',
         F5.1,10X,
         +'*** DO SDVIGA***)
         CALL PRINT2(MMPU,MF1,MP2,MF3,MP4,MU,MC,N,S)
```

Shift of Matrices MF1, MP2, MF3, MP4

```
  C***   BLOCK 19***OBLIQUE SHIFT OF THE MATRICES MF1,MP2,MF3,MP4
 2000    W=TAU/H
         W=W+0.5
         KR=W
         SI=S+0.5
         ISR=S1
         IN=ISR+1
         M=KR*(ISR+1)
         CALL ZERNO(MF1,IN,M,ISR,KR)
         CALL ZERNO(MP2,IN,M,ISR,KR)
         CALL ZERNO(MF3,IN,M,ISR,KR)
         CALL ZERNO(MP4,IN,M,ISR,KR)
```

Shift of the Matrices MC and MU

```
  S***   BLOCK 22***SHIFT OF THE MATRIX MC
         CALL PERNO(MC,8,M,ISR,KR)
  C***   BLOCK 24 CDBIG MATP. MU
         CALL PERNO(MU,2,M,ISR,KR)
         CALL PERNO(MMPU,4,M,ISR,KR)
         MMPU(1,ISTH)=MF1(1,KSTH)
         MMPU(2,KSTH)=MP2(1,KSTH)
         MMPU(3,KSTH)=MF3(1,KSTH)
```

Formation of the Matrix of Initial Conditions of the Model

```
         MMPU(4,KSTH)=MP4(1,KSTH)
  C***   BLOCK 25***FILLING OF THE MATRIX MHY
         NSTB=TAUC4/TAU+1
         LKON=S+1,5
         MHY(1,NSTB)=MF1(1,KSTH)
         MHY(2,NSTB)=MP2(1,KSTH)
         MHY(3,NSTB)=MF3(1,KSTH)
         MHY(4,NSTB)=MP4(1,KSTH)
         MHY(5,NSTB)=MU(1,KSTH)
         MHY(6,NSTB)=MU(2,KSTH)
         WRITE(99,790)((MHY(I,K),I=1,6),K=1,LKON)
  790    FORMAT('',10X,'MATRIX MHY (BY COLUMNS)'//('',6F11.6))
```

```
C***    BLOCK 26
        BW=RO*(S+1.0)*TAU
        IF(TAUC4,GE,BW)GO TO 10000
C***    BLOCK 27
        TAUC4=TAUC4+TAU
        GO TO 10007
10000   STOP
        END
```

Formation of the Algorithm Memory

```
        SUBROUTINE PERNO(MASS,N,M,S,K)
        REAL MASS(N,M)
        INTEGER S,K,PAR
        DO 20 PAR=1,S
        DO 20 I=1,N
        K1=1+(PAR=1)*K
        K2=K+(PAR=1)*K
        DO 10 J=K1,K2
        J1=J+K
10      MASS(I,J)=MASS(I,J1)
20      CONTINUE
        N2=M=K+1
        DO 40 I=1,N
        DO 40 J=N2, M
40      MASS(I,J)=0.0
        RETURN
        END
```

Auxiliary Programs (Printout of the Files)

```
        SUBROUTINE PRINT2(MMPU,MF1,MP2,MF3,MP4,MU,MC,N,S)
        REAL MF1(5,25),MP2(5,25),MF3(5,25),MP4(5,25)
        REAL MU(2,N),MC(8,N)
        REAL MMPU(4,N)
        IS1=S+1.4
        WRITE(99,80)
80      FORMAT(//'',10X,'MATRIX MMPU')
        WRITE(99,20) ((MMPU(I,K),K=1,10),I=1,4)
        WRITE(99,20)((MMPU(I,K),K=11,20),I=1,4)
        WRITE(99,21)((MMPU(I,K),K=21,25),I=1,4)
        GO TO 100
        WRITE(99,10)
10      FORMAT(//'',10X,'MATRIX MF1')
        WRITE(99,20)((MF1(I,K),K=1,10),I=1,IS1)
        WRITE(99,20)((M H(I,K),K=11,20),I=1,IS1)
        WRITE(99,21)((MF1(I,K),K=21,25),I=1,IS1)
20      FORMAT(/'',10(1X,F11.6))
21      FORMAT(/'',5(1X,F11.6)
        WRITE(99,30)
```

```
 30   FORMAT(//'',10X,'MATRIX MP2')
      WRITE(99,20)((MP2(I,K),K=1,10),I=1,IS1)
      WRITE(99,20)((MP2(I,K),K=11,20),I=1,IS1)
      WRITE(99,21)((MP2(I,K),K=21,25),I=1,IS1)
      WRITE(99,40)
 40   FORMAT(//'',10X,'MATRIX MF3')
      WRITE(99,20)((MF3(I,K),K=1,10),I=1,IS1)
      WRITE(99,20)((MF3(I,K),K=11,20),I=1,IS1)
      WRITE(99,21)((MF3(I,K),K=21,25),I=1,IS1)
      WRITE(99,50)
 50   FORMAT(//'',10X,'MATRIX MP4')
      WRITE(99,20)((MP4(I,K),K=1,10),I=1,IS1)
      WRITE(99,20)((MP4(I,K),K=11,20),I=1,IS1)
      WRITE(99,21)((MP4(I,K),K=21,25),I=1,IS1)
 100  WRITE(99,60)
 60   FORMAT(//'',10X,'MATRIX MU')
      WRITE(99,20)((MU(I,K),K=1,10),I=1,2)
      WRITE(99,20)((MU(I,K),K=11,20),I=1,2)
      WRITE(99,21)((MU(I,K),K=21,25),I=1,2)
      WRITE(99,70)
 70   FORMAT(//'',10X,'MATRIX MC')
      WRITE(99,20)((MC(I,K),K=1,10),I=1,8)
      WRITE(99,20)((MC(I,K),K=11,20),I=1,8)
      WRITE(99,21)((MC(I,K),K=21,25),I=1,8)
      RETURN
      END
      SUBROUTINE WHODM(F,N,X,Y)
      REAL*8F,X,Y(15),ALFA(6,6),FMD(6)
      GO TO 100
      ENTRY PR4MOD(ALFA,FMD)
      RETURN
 100  F=0.0D0
      DO 10J=1,6
 10   F=F+ALFA(N,J)*Y(J)
      F=F+FMD(N)
      RETURN
      END
      SUBROUTINE FMODF (F,N,X,Y)
      REAL*8F,X,Y(15),ALFA(6,6),FMD(6),USYS,USYSO,USYS1,
      XAD,HD
      GO TO 100
      ENTRY FMOD(ALFA,FMD,USYSO,USYS1,XAD,HD)
      RETURN
 100  F=0.0D0
      USYS=USYSO+(X=XAD)*(USYS1=USYS))/HD
      DO 10J=1,4
 10   F=F+ALFA(N,J)*Y(J)
      F=F+FMD(N)+ALFA(N,5)*USYS
      RETURN
      END
```

```
      SUBROUTINE WHODC(F,N,X,Y)
      REAL*8F,X,Y(15), APRC(8,8), BPRC(8), FPRC(8), US?S,
      USYS0, USYS1,
      +XAD,HD
      GO TO 100
      ENTRY PR4SYS(APRC,BPRS,FPRS,USYS0,USYS1,XAD,HD)
      RETURN
100   F=0.0D0
      DO 10 J=1,8
      USYS=USYS0+(X=XAD)*(USYS1=USYS0)/HD
10    F=F+APRC(N,J)*Y(J)
      F=F+BPRC(N)*USYS+FPRC(N)
      RETURN
      END
```

Example. Let the object of control be given by the equation (3.20), in Chapter 3 where

$$A=\begin{pmatrix} 10 & -1,0 & -1,0 & -1,0 \\ 0 & 1,0 & 0,102 & 0,765 \\ -0,1 & 0,007 & 1,0 & 0 \\ -0,138 & 0,073 & 0 & 1 \end{pmatrix};\ B=\begin{pmatrix} 0 & -0,005 & 0 & 0 \\ 0 & -0,007 & 0 & 0 \\ 0 & 0 & -0,003 & 0 \\ 0 & 0 & 0 & -0,002 \end{pmatrix};$$

$$C=\begin{pmatrix} 0 & -4,6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6,2 & 0 \\ 0 & 0 & 0 & -7,13 \end{pmatrix};\ \vec{D}=\begin{pmatrix} -0,044 \\ -0,385 \\ 0 \\ 0 \end{pmatrix};\ \vec{F}=\begin{pmatrix} 0,05 \\ 0,01 \\ 0 \\ 0 \end{pmatrix};$$

$$s_1(0)=0,001;\quad s_2(0)=0,002.$$

The remaining components of the initial vector are equal to zero.

The calculation with respect to the function $W(p^2)=\Phi_k(p^2)/\Phi_0(p^2)$ indicates that the investigated object is structurally unstable and naturally dynamically stable.

The equation of the control system is taken in the form

$$0,015\ddot{u}+0,2\dot{u}+u=10,8\psi+9,2\dot{\psi}-0,02z-0,01\dot{z}.$$

The remaining control parameters have the following numerical values: T=6 sec (length of the analysis segment); τ=1.4 sec (length of the adjustment segment); s=4 (number of adjustments of the model in the analysis segment).

The transition processes in the investigated system have the form indicated in Fig 4.17, where the behavior in time of the generalized coordinates u(t), ψ(t), respectively, is depicted.

The investigated example has the purpose of noting two facts:

The process of adaptation of the model to external disturbances (the step curve in Fig.4.16, b);

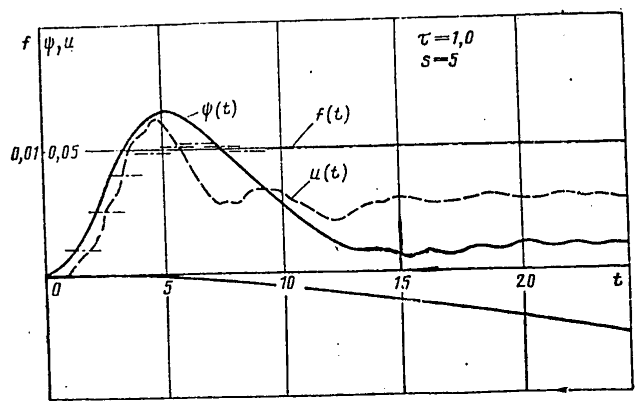

Fig 4.17. Typical nature of the transient processes in the stabilization mode of the space vehicle with a discrete model

Possibility of stabilization of a structurally unstable object using an adaptive model with relatively small damping coefficients;

$$\varepsilon_1 = 0{,}003;\ \varepsilon_2 = 0{,}002.$$

As is obvious from Fig 4.17, the closed system is stable, the transient process in practice ends after 12 seconds. The characteristics of the transient process can be improved when necessary by selecting the values of the controlling constants.

For comparison let us calculated the required values of coefficients ε_1 insuring stability of the closed system for the "classical" method of control. Using formulas (1.55), we find that $\varepsilon \cong \varepsilon_2 = 0.1$.

It is possible to insure the corresponding damping by the existing means only by substituting quite powerful (in the sense of mass) baffles in the tanks with the liquid fuel, which frequently is undesirable.

The application of digital filters therefore presents additional possibilities for attenuating the effect of the structural instability in cases where the composition of the object turns out to be insufficiently successful.

It is essential that in the proposed theory and the investigated examples the stabilizability criterion as characteristic of the possibility of phase stabilization of the space vehicle does not depend on the specific parameters of the control system, but is determined only by the characteristic parameters of the system considered as the object of control.

This fact also determines the possibility of its use in the initial steps of design of the space vehicle under the conditions of incompleteness of information about the stabilization system (the control system is created, as a rule, for a "finished" object).

As is obvious, not all of the compositional layouts of the space vehicle are identically favorable from the point of view of dynamics. For example, the use of toroidal (in general, biconnected) fuel tanks in practice exclude stabilization of the space vehicle in the active segment without structural modification of the fuel tanks (installation of oscillation dampers) or complicated of the stabilization system.

It is known that the alignment of the space vehicle, being by definition a static characteristic of the object, essentially also influences its stabilizability. The objects with upper or lower alignment are, as a rule, more improved in the dynamic sense than with intermediate alignment.

The unfavorable dynamic properties of a number of compositional layouts of space vehicles (natural and structural instability) require additional measures both when developing the object directly (the space vehicle directly) and when developing the control system. The analysis presented in sections 4.3 to 4.4 permits formulation of these requirements on the part of selecting both the damping coefficients and the parameters of the automatic stabilization system. In a numb er of cases these requirements are far from simple and obviously require theoretically new solutions, for example, the development of other stabilization principles.

CHAPTER 5. STABILIZABILITY AND STABILITY OF SPACE VEHICLES IN THE ACTIVE SEGMENT OF FLIGHT

Approximate Method of Investigating the Stability of Space Vehicles. Amplitude and Phase Stabilization

Conversion of the Equations of Motion of a Space Vehicle

The system of equations (3.24) of the controlled movement of a space vehicle in the active segment is written in the following form:

$$A\ddot{\vec{x}}+\varepsilon B\dot{\vec{x}}+C\vec{x}=\vec{b}u;$$
$$u=Z_1(\psi^0)-L_2(z^0), \qquad (5.1)$$

where

$$\vec{x}=(z,\ \psi,\ s_1,\ldots,\ s_n,\ q_1,\ldots,\ q_m);$$

$$\vec{\xi}_\psi=(0,\ \psi,\ 0,\ldots,\ 0,\ q_1,\ldots,\ q_m),\ \vec{\xi}_z=(z,\ \psi,\ 0,\ldots,\ 0,\ q_1,\ldots,\ q_m)$$

are the vectors of state and observation of the system (5.1);

$$\vec{b}=(-a_{zu},\ -a_{\psi u},0\ ,\ldots,\ 0,\ -a_{q_1u},\ldots,\ -a_{q_mu});$$

$$\vec{g}_\psi=(0,\ 1,\ 0,\ldots,\ 0,\ -\eta_1',\ldots,\ -\eta_m');$$

$$\vec{g}_z=(0\ \ 1,\ -(x^0-x_c),\ 0,\ldots,\ 0,\ \eta_1',\ldots,\ \eta_m')$$

are the constant vecto.:s ٦f the same dimensionality (n=k+m+2).

On the basis of equations (3.24) the matrices A, B and C have the form

$$\underset{n\times n}{A}=\begin{pmatrix} 1 & 0 & a_{zs_1} & \ldots & a_{zs_k} & 0 & \ldots & 0 \\ 0 & 1 & a_{\psi s_1} & \ldots & a_{\psi s_k} & 0 & \ddots & 0 \\ a_{s_1z} & a_{s_1\psi} & 1 & \vdots & 0 & a_{s_1q_1} & \ldots & a_{s_1q_m} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{s_kz} & a_{s_k\psi} & 0 & \ldots & 1 & a_{s_kq_1} & \ldots & a_{s_kq_n} \\ 0 & 0 & a_{q_1s_1} & \vdots & a_{q_1s_k} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{q_ms_1} & & a_{q_ms_k} & 0 & & 1 \end{pmatrix};$$

$$\underset{n\times n}{C}=\begin{pmatrix} 0 & -a_{z\psi} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & -\sigma_1^2 & \vdots & 0 & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \vdots & -\sigma_k^2 & -\omega_1^2 & \vdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & -\omega_m^2 \end{pmatrix};$$

$$B=\mathrm{diag}(0,\ 0,\ -\varepsilon_{s_1},\dots,\ -\varepsilon_{s_k},\ -\varepsilon_{q_1},\dots,\ -\varepsilon_{q_m}).$$

Let us reduce the equations (5.1) to normal form

$$\ddot{\vec{x}}+\varepsilon B_{\text{н}}\dot{\vec{x}}+C_{\text{н}}\vec{x}=\vec{b}_{\text{н}}u; \qquad (5.2)$$
$$u=L_1(\psi^0)-L_2(z^0),$$

calculating the matrices

$$B_{\text{н}}=A^{-1}B;\ C_{\text{н}}=A^{-1}C;\ b_{\text{н}}=A^{-1}b.$$

Considering the smallness of the cross relations in matrix A and also the structure of the matrix B, in the transformed matrix B_H let us retain only the diagonal terms:

$$B_{\text{н}}=\mathrm{diag}(0,\ 0,-a_{s_1}\varepsilon_{s_1},\dots,\ -a_{s_k}\varepsilon_{s_k},\ -a_{q_1}\varepsilon_{q_1},\dots,\ -a_{q_m}\varepsilon_{q_m}), \qquad (5.3)$$

where a_{s_i}, a_{q_i} are the diagonal terms of matrix A^{-1} corresponding to the two investigated groups of oscillatory elements:

$$(s_1,\dots,\ s_k)\ \&\ (q_1,\dots,\ q_m).$$

Let us assume that the roots of the characteristic equation

$$|C_{\text{н}}-\lambda^2E|=0 \qquad (5.4)$$

are real and different:

$$\lambda_z=-\omega_z^2;\ \lambda_\psi=-\omega_\psi^2;\ \lambda_{s_i}=-\tilde{\sigma}_i^2,\ \lambda_{q_j}=-\tilde{\omega}_j^2;$$
$$i=1,\ 2,\dots,\ k;\ j=1,\ 2,\dots,\ m.$$

In this case a nondegenerate transformation exists

$$\vec{x}=T\vec{y}:$$

$$T=\left(\begin{array}{cc|ccc|ccc} t_{zz} & t_{z\psi} & t_{zs_1} & \dots & t_{zs_k} & t_{zq_1} & \dots & t_{zq_m} \\ t_{\psi z} & t_{\psi\psi} & t_{\psi s_1} & \dots & t_{\psi s_k} & t_{\psi q_1} & \dots & t_{\psi q_m} \\ \hline t_{s_1z} & t_{s_1\psi} & t_{s_1s_1} & \dots & t_{s_1s_k} & t_{s_1q_1} & \dots & t_{s_1q_m} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & & \vdots \\ t_{s_kz} & t_{s_k\psi} & t_{s_ks_1} & \dots & t_{s_ks_k} & t_{s_kq_1} & \dots & t_{s_kq_m} \\ \hline t_{q_1z} & t_{q_1\psi} & t_{q_1s_1} & \dots & t_{q_1s_k} & t_{q_1q_1} & \dots & t_{q_1q_m} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ t_{q_mz} & t_{q_m\psi} & t_{q_ms_1} & \dots & t_{q_ms_k} & t_{q_mq_1} & \dots & t_{q_mq_m} \end{array}\right); \qquad (5.5)$$

$$t_{\psi\psi}=t_{zz}=1;\ t_{s_is_i}=1;\ i=1,\ 2,\dots,\ k;\ t_{q_jq_j}=1,\ j=1,\ 2,\dots,\ m,$$

reducing system (5.2) to canonical form

$$\dot{\vec{y}} = \Lambda\vec{y} + \varepsilon B_k \dot{\vec{y}} + \vec{b}_k u;$$
$$u = L_1(\psi_y^0) - L_2(z_y^0);\ \psi_y^0 = (\vec{q}_\psi, \vec{\xi}_\psi)\ \psi_z^0 = (\vec{q}_z, \vec{\xi}_\psi), \quad (5.6)$$

where

$$\Lambda = \mathrm{diag}\,(-\omega_z^2, -\omega_\psi^2, -\bar{\sigma}_1^2, \ldots, -\bar{\sigma}_k^2, -\bar{\omega}_1^2, \ldots, -\omega_m^2),\ \vec{b}_k = T^{-1}\vec{b}_k,$$

and $\vec{\xi}_\psi$, $\vec{\xi}_z$ are the observation vectors of the system (5.1) calculated on a new base. The expressions for $\vec{\xi}_\psi$, $\vec{\xi}_z$ are more specifically defined below.

The further problem consists in investigating the canonical system (5.6) in order to find the conditions of stability of the space vehicle in the oscillation frequency range of the fuel in the compartments $\{\sigma_i\}$ and the elastic oscillations in the structure $\{\omega_i\}$. Here it is desirable to consider the mutual effect of the corresponding partial systems on each other and also the influence of certain additional factors.

Effect of the Fuel Mobility in the Compartments on the Stability of a Space Vehicle

Let us exclude the z-coordinate and the generalized coordinate $q_1, \ldots, q_m$ from the investigation. The canonical system (5.6) will be obtained in the form

$$\ddot{\vec{y}} = \Lambda\vec{y} + \varepsilon B_k \ddot{\vec{y}} + \vec{b}_k u,$$
$$u = L(v), \quad (5.7)$$

where

$$\Lambda = \mathrm{diag}\,(-\omega_z^2, -\omega_\psi^2, -\bar{\sigma}_1^2, \ldots, -\bar{\sigma}_k^2);\ \vec{b}_k = T^{-1}\vec{b};$$
$$U = \psi_y^0 = (\vec{\tau}, \vec{y}),\ \vec{\tau} = (1, t_{\psi s_1}, \ldots, t_{\psi s_k}),$$

and the formulas relating the initial (x) and canonical (y) variables have the following form on the basis of expression (5.5)

$$\psi = y_\psi + t_{\psi s_1} y_1 + \ldots + t_{\psi s_k} y_k;$$
$$s_1 = t_{s_1\psi} y_\psi + y_1 + \ldots + t_{s_1 s_k} y_k; \quad (5.8)$$
$$\cdots\cdots\cdots\cdots$$
$$s_k = t_{s_k\psi} y_\psi + t_{s_k s_1} y_1 + \ldots + y_k.$$

Let the following representation be admissible for the operator L(v) in the frequency range of $\bar{\sigma}_i^2$ (i=1,2,...,k):

$$L(v) = L(p)\,v = (k_0 + k_1 p)\,v, \quad (5.9)$$

where

$$k_0 = (\sigma_i) = A(\sigma_i)\cos\varphi(\sigma_i);\ k_1(\sigma_i) = \frac{A(\sigma_i)}{\sigma_i}\sin\varphi(\sigma_i).$$

In order to estimate the mutual effect of the partial systems corresponding to the indexes k, k+1, on the one hand, and the partial system corresponding to the index ψ (the angular movement of the system), on the other, let us consider instead of expression (5.7) the simplified system

$$\begin{aligned} &\ddot{y}_\psi = \lambda_\psi y_\psi + \beta_{\psi u} u; \\ &\ddot{y}_1 = -\sigma_1^2 y_1 + \beta_{s_1 u} u; \qquad v = \tau' y; \\ &\ddot{y}_2 = -\sigma_2^2 y_2 + \beta_{s_2 u} u; \\ &u = L(p) v, \end{aligned} \tag{5.10}$$

where the dissipative terms are omitted.

For simplification of the notation it is also accepted that σ_1 and then ω_1 now denote the natural frequencies of the system (5.1) and not the partial ones as was assumed in the initial system (5.1).

The characteristic equation of the system (5.10) considering the expression (5.9) will be represented in the form

$$\begin{vmatrix} p^2 - (k_0 + k_1 p)\chi & a_1\sigma_1^2 & a_2\sigma_2^2 \\ -(k_0 + k_1 p)\beta_{s_1 u} & p^2 + \sigma_1^2 & 0 \\ -(k_0 + k_1 p)\beta_{s_2 u} & 0 & p^2 + \sigma_2^2 \end{vmatrix}, \tag{5.11}$$

where $\chi = \beta_{\psi u} + t_{\psi s_1}\beta_{s_1 u} + t_{\psi s_2}\beta_{s_2 u};\ a_1 = t_{\psi s_1},\ a_2 = t_{\psi s_2}$.

Let us set

$$p^2 = -\sigma_1^2 + \Delta p. \tag{5.12}$$

Then with accuracy to the terms of second order smallness

$$p = i\sigma_1 - i\frac{\Delta p}{2\sigma_1}. \tag{5.13}$$

Substituting expressions (5.12) and (5.13) in the determinant (5.11), in the first approximation we obtain:

$$\begin{aligned} \Delta p \Bigg[&-\sigma_1^2 - k_1\varkappa + \frac{a_1\sigma_1^2}{\sigma_2^2 - \sigma_1^2} k_1\beta_{s_1 u} + \frac{a_2\sigma_2^2}{\sigma_2^2 - \sigma_1^2} k_1\beta_{s_2 u} + i\sigma_1 k_0\varkappa - \\ &- \frac{k_0 a_1 \sigma_1^2 \beta_{s_1 u}}{2\sigma_1} + \frac{a_1\sigma_1^2 k_0 \sigma_1 \beta_{s_1 u}}{\sigma_2^2 - \sigma_1^2} + \frac{a_2\sigma_2^2 k_0\sigma_1\beta_{s_1 u}}{\sigma_2^2 - \sigma_1^2}\Bigg] + a_1\sigma_1^2 k_1 \beta_{s_1 u} + \\ &+ i a_1 \sigma_1^2 k_1 \sigma_1 \beta_{s_1 u} = 0. \end{aligned} \tag{5.14}$$

Setting $\Delta p = \beta + i\alpha$, for determination of the values of β and α we obtain the following system of equations:

$$\begin{cases} \alpha\left(-1+\frac{k_0}{\sigma_1^2}\right)+\beta\frac{k_1}{\sigma_1}\left(Q-\frac{a_1\beta_{s_1u}}{2}\right)+a_1k_1\sigma_1\beta_{s_1u}=0; \\ \beta\left(-1+\frac{k_0}{\sigma_1^2}\right)-\alpha\frac{k_1}{\sigma_1}\left(Q-\frac{a_1\beta_{s_1u}}{2}\right)+a_1k_0\beta_{s_1u}=0, \end{cases} \quad (5.15)$$

where

$$Q=-\varkappa+\frac{a_1\sigma_1^2}{\sigma_2^2-\sigma_1^2}\beta_{s_1u}+\frac{a_2\sigma_2^2}{\sigma_2^2-\sigma_1^2}\beta_{s_2u}.$$

Finally, the formulas for calculating the real and imaginary parts of the root p_j with the imaginary parts closest to the partial frequency σ_j assume the form

$$\operatorname{Re} p_1=\frac{\alpha}{2\sigma_1};\quad \operatorname{Im} p_1=\frac{\beta}{2\sigma_1}. \quad (5.16)$$

The values of α and β are solutions of the linear system of equations (5.15) in which the parameters

$$k_1(\omega)=A(\omega)\cos\varphi(\omega);\ k_0(\omega)=\frac{1}{\omega}A(\omega)\sin\varphi(\omega)$$

are calculated for values of $\omega=\sigma_1$.

The formulas for calculating the real part of Re_{p_2} have analogous form; it is only necessary to calculate the parameters $B(\omega)$, $A(\omega)$ in the equation (5.15) for values of $\omega=\sigma_2$.

In a number of cases the system of equations (5.15) permits some simplification. For example, let us assume that the connectedness of the partial systems corresponding to the frequency σ_1 and σ_2 is small, that is, the following inequalities are satisfied

$$\left|\frac{a_1\sigma_1^2}{\sigma_2^2-\sigma_1^2}\beta_{s_1u}\right|\leqslant 1\left|\frac{a_2\sigma_2^2}{\sigma_2^2-\sigma_1^2}\beta_{s_2u}\right|\ll 1. \quad (5.17)$$

In formulas (5.15) let us set $Q=-\chi$. We find

$$\begin{aligned} &\alpha\left(1+\frac{\varkappa}{\sigma_1^2}\right)+\beta\left(\frac{A\varkappa}{\sigma_1^2}+\frac{Aa_1\beta_{s_1u}}{2\sigma_1}\right)=a_1A\sigma_1\beta_{s_1u}; \\ &-\alpha\left(\frac{A\varkappa}{\sigma_1^2}+\frac{Aa_1\beta_{s_1u}}{2\sigma_1}\right)+\beta\left(1+\frac{B\varkappa}{\sigma_1^2}\right)=a_1B\beta_{s_1u} \end{aligned} \quad (5.18)$$

Then let the connectedness of the partial solid state (ψ)-liquid (s_1) systems be small:

$$\frac{B\varkappa}{\sigma_i^2}\ll 1;\ \left|\frac{A\varkappa}{\sigma_i}+\frac{Aa_i\beta_{s_iu}}{2\sigma_i}\right|\ll 1. \quad (5.19)$$

The formulas (5.17) for calculating α and β, $\overline{i=1,2}$ a e greatly simplified and assume the form

$$\alpha_i = a_i A \sigma_i \beta_{s_i u}; \quad \beta_i = a_i B \beta_{s_i u}.$$

The expressions for Re p_i, Im p_i vary correspondingly:

$$\operatorname{Re} p_i = \frac{a_i A(\sigma_i)\beta_{s_i u}}{2}; \quad \operatorname{Im} p_2 = -\frac{a_i B(\sigma_i)\beta_{s_i u}}{2\sigma_i} \quad (i=1,\ 2). \quad (5.20)$$

Since

$$A(\sigma_i) = \frac{1}{\sigma_i}\operatorname{Im}[L(i\sigma)],$$

then from the formulas (5.18) it is clear that the stability of the space vehicle in the frequency range of the oscillations of the fuel essentially depends on the phase shift of the regulator L(v) on these frequencies.

Example. Let the numerical values for the coefficients of the system of equations of disturbed motion of the space vehicle (1.6) for i=1,2 be as follows:

$$a_{zs_1} = -0{,}0152; \quad a_{s_1 z} = -1{,}2650; \quad \sigma_1^2 = 32{,}810;$$
$$a_{zs_2} = -0{,}2202; \quad a_{s_1\psi} = -2{,}0980; \quad \sigma_2^2 = 41{,}1210;$$
$$a_{\psi s_1} = -0{,}0056; \quad a_{s_2 z} = -1{,}500; \quad a_{z\psi} = -15{,}580;$$
$$a_{\psi s_2} = +0{,}0099; \quad a_{s_2\psi} = 3{,}0530; \quad a_{zu} = -2{,}7630;$$
$$a_{\psi u} = -3{,}0010.$$

The automatic stabilization system of the space vehicle is given by the equation

$$0{,}0098\ddot{u} + 0{,}1260\dot{u} + u = 10{,}50\psi + 6{,}30\dot{\psi},$$

so that in the given case

$$W(p) = \frac{10{,}50 + 6{,}30p}{0{,}0098p^2 + 0{,}126p + 1}.$$

The coefficients of the canonical form of (5.10) are as follows:

$$\sigma_1^2 = 33{,}970; \quad a_1 = t_{\psi s_1} = 0{,}0059; \quad \beta_{\psi u} = -0{,}3010;$$
$$\sigma_2^2 = 43{,}650; \quad a_2 = t_{\psi s_2} = 0{,}0098; \quad \beta_{s_1 u} = -1{,}1775;$$
$$\beta_{s_2 u} = -0{,}3055.$$

Let us present some intermediate results:

$$k_1(\sigma_1) = 2{,}930; \quad k_0 = 34{,}4580; \quad \sigma_1 = 5{,}8280;$$
$$k_1(\sigma_2) = 2{,}239; \quad k_0 = 39{,}7920; \quad \sigma_2 = 6{,}6070;$$

The coefficients of the system (5.15):

$$1+\frac{k_0}{\sigma_1^2}=0,6350;\quad \frac{k_1\varkappa}{\sigma_1}+\frac{k_1\sigma_1\beta_{s_1u}}{2\sigma_1}=0,120;$$
$$a_1k_1\sigma_1\beta_{s_1u}=-0,0908;\ a_1k_1\beta_{s_1u}=-0,2770.$$

Table 5.1

(1) Корни	(2) Точное значение	(3) Формула (5.15)	(4) Формула (5.20)
p_1	$-0,0193\pm5,8535i$	$-0,0187\pm5,8620i$	$-0,0102\pm5,9480i$
p_2	$-0,0066\pm6,6187i$	$-0,0064\pm6,6180i$	$-0,0033\pm6,7260i$

Key:
1. Roots
2. Exact value
3. Formula (5.15)
4. Formula (5.20)

The results of calculating the roots are presented in Table 5.1.

Comparing the exact (calculated on the digital computer) and approximate values of Re p_i (i=1,2) calculated by formula (5.15), it is possible to see that the error in the given case does not exceed 5%.

Then it is necessary to discuss the relation of the structural stability of the space vehicle and its stability understood in the classical sense.

Let the conditions (5.17), (5.19) be satisfied approximately. Then in the general case, on the basis of the expression (5.20)

$$a_i=t_{\psi s_i}\beta_{s_iu}\sigma_ik_i(\sigma_i).$$

The criterion of structural stability of the space vehicle in the given case applied to the system of two oscillators s_1 and s_2 has the form

$$\operatorname{sign}(t_{\psi s_1}\beta_{s_1u})=\operatorname{sign}(t_{\psi s_2}\beta_{s_2u}). \qquad (5.21)$$

In the given case, since

$$t_{\psi s_1}\beta_{s_1u}=-0,18\cdot10^{-2}<0 \quad t_{\psi s_2}\beta_{s_2u}=-0,30\cdot10^{-2}<0,$$

the condition (5.21) is satisfied and, consequently, for stability of the space vehicle on the investigated frequencies σ_1, σ_2, the phase lead of the automatic stabilization system on the following frequencies is required

$$\sin\varphi(\sigma_1)>0;\quad \sin\varphi(\sigma_2)>0. \qquad (5.22)$$

Since for the given control system $k_1(\sigma_1)=2.931>0$; $k_1(\sigma_2)=2.239$, the condition (5.22) is also satisfied.

Consequently, the noncontradictory nature of the phase conditions for the object of control on frequencies of σ_1, σ_2 [the equality (5.21) -- structural stability of the space vehicle on the frequencies] in the given case also indicates (on the basis of expression (5.22)) its stability in the classical sense.

Consideration of Elasticity of the Space Vehicle Structure

The canonical system (5.6) of equations of motion of the space vehicle will be written in the form

$$\begin{aligned} \ddot{y}_\psi &= \lambda_\psi y_\psi + \beta_{\psi u} u; \\ \ddot{y}_{s_i} &= -\sigma_i^2 y_{s_i} + \beta_{s_i u} u; \\ \ddot{y}_{q_j} &= -\omega_j^2 y_{q_j} + \beta_{q_j u} u; \\ v &= \vec{\tau}', \vec{y}; \\ u &= L(p)\bar{v}, \end{aligned} \qquad (5.23)$$

where

$$\tau = (1 - \eta_j' t_{q_j \psi},\ t_{\psi s_i} - \eta_j' t_{q_j} s_i, \ldots,\ t_{\psi q_j} - \eta_j').$$

The calculations analogous to those presented above lead to the following approximate expressions for disturbances of the natural values of σ_i^2, ω_j^2 caused by the effect of the control system (the automatic stabilization system):

$$\begin{aligned} \operatorname{Re} p_{s_i} &= \frac{(t_{\psi s_i} - \eta_j' t_{q_j s_i})\, \beta_{s_i u} k_1(\sigma_i)}{2\sigma_i}; \\ \operatorname{Im} p_{s_i} &= \frac{(t_{\psi s_i} - \eta_j' t_{q_j s_i})\, \beta_{s_i u} k_0(\sigma_i)}{2} \end{aligned} \qquad (5.24)$$

in the frequency band of the oscillations of the liquid filling the tanks and

$$\begin{aligned} \operatorname{Re} p_{q_j} &= \frac{(t_{\psi q_i} - \eta_j')\, \beta_{q_j u} k_0(\omega_j)}{2\omega_j}; \\ \operatorname{Im} p_{q_j} &= \frac{(t_{\psi q_j} - \eta_j')\, \beta_{q_j u} k_0(\omega_j)}{2} \end{aligned} \qquad (5.25)$$

in the frequency band of the elastic vibrations of the structure where, just as before, we have the following notation

$$k_0(\Omega) = A(\Omega)\cos\varphi(\Omega);\ k_1(\Omega) = \frac{A(\Omega)}{\Omega}\sin\varphi(\Omega).$$

The coefficients entering into expression (5.24), (5.25) have the following physical meaning:

$t_{\psi s_i}$ are the coefficients characterizing the connectedness of the partial solid state-liquid systems (i=1,2,...,k);

$t_{\psi q_j}$ are the coefficients characterizing the connectedness of the partial systems corresponding to the oscillations of the space vehicle as a solid state and the elastic vibrations of the space vehicle structure (j=1,2,...,m);

$t_{q_j s_i}$ are the coefficients characterizing the connectedness of the partial systems, one of which corresponds to the j-th form of the elastic vibrations of the space vehicle hull, and the other, the basic body of the fuel oscillations in the i-th compartment;

n_j' is the derivative form of the elastic vibrations of the space vehicle structure at the point of installation of the measuring elements (the gyroscopic angle cage). The following situations are of practical interest:

$$1. \quad |t_{\psi s_i}| \gg |\eta_j' t_{q_j s_i}| \quad (i=1,\ 2,\ldots,\ k). \tag{5.26}$$

Simplifying the formulas (5.24), we find

$$\operatorname{Re} p_{s_i} = \frac{t_{\psi s_i} \beta_{s_i u} k_1(\sigma_i)}{2\sigma_i}, \quad \operatorname{Im} p_{s_i} = \frac{t_{\psi s_i} \beta_{s_i u} k_1(\sigma_i)}{2}.$$

and, as is obvious, we obtain the same relations as in the preceding section. Thus, the condition (5.26) can be used as the criterion of applicability of the simplified canonical system (5.10).

In the given case it is possible to neglect the elasticity of the space vehicle hull. The stability of the space vehicle in the frequency range of oscillations of the fuel is basically determined by the connectedness of the partial space vehicle systems as solid state-liquid systems.

$$2. \quad |t_{\psi s_i}| \sim |\eta_j' t_{q_j s_i}|.$$

The stability of a space vehicle in the frequency band of the fuel oscillations is determined by the connectedness of the partial solid state-liquid systems -- the hull elasticity. For the calculations it is necessary to use the formulas (5.24) or more exact expressions presented at the end of the section.

$$3. \quad |t_{\psi s_i}| \ll |\eta_j' t_{q_j s_i}|.$$

Simplifying formulas (5.24), we find

$$\operatorname{Re} p_{s_i} = -\frac{\eta_j' t_{q_j s_i} \beta_{s_i u} k_1(\sigma_i)}{2}; \quad \operatorname{Im} p_{s_i} = -\frac{\eta_j' t_{q_j s_i} \beta_{s_i u} k_0(\sigma_i)}{2}.$$

In this case the signs of the expressions

$$t_{\psi s_i} \,\&\, (-\eta_j' t_{q_j s_i})$$

do not coincide, the effect of the variation of the structural properties of the object of control in the frequency band of the oscillations of the liquid filling the compartments caused by the effect of elasticity of the space vehicle structure is observed. The possibility of this effect essentially follows from the example in Chapter 3: it is required that the parameters of the space vehicle lie within the limits corresponding to the crosshatched regions in Fig 3.25.

Thus, the effect of the j-th form of the oscillations of the space vehicle structure on its stability in the vicinity of the frequency of the principal tone of the fuel oscillations in the i-th tank is determined by the value of the parameter

$$\varkappa_{ji} = \eta_j' t_{q_j s_i} / t_{\psi s_i},$$

where the following values correspond to cases 1-3:

$$\varkappa_{ji} \ll 1; \quad \sigma_{ji} \sim 1; \quad \varkappa_{ji} \gg 1.$$

The reverse effect ("liquid" on "elasticity") is small, as the numerical analysis shows, and for calculation of the corrections to the eigenvalues of the system in the oscillation frequency range of the hull in the first approximation is possible to use the expressions

$$\operatorname{Re} p_{q_j} = -\frac{\eta_j' \beta_{q_j u} k_1(\omega_j)}{2\omega_j}; \quad \operatorname{Im} p_{q_j} = \frac{\eta_j' \beta_{q_j u} k_0(\omega_j)}{2}.$$

Based on expression (5.24)-(5.25), let us consider the problem of the "amplitude" and "phase" stabilization of the space vehicle.

From expressions (5.24)-(5.25), we have the conditions of stability of a closed system:

$$(t_{\psi s_i} - \eta_j' t_{q_j s_i})\, \beta_{s_i u} k_1(\sigma_i) < 0 \qquad (5.27)$$

in the frequency range of oscillations of the liquid filling the tanks;

$$(t_{\psi q_j} - \eta_j')\, \beta_{q_j u} k_1(\omega_j) < 0 \qquad (5.28)$$

in the range of frequencies of other vibrations of the spacecraft hull.

Let us write the equivalent of the conditions of structural stability for the systems (5.1):

$$\operatorname{sign}\left[(t_{\psi s_i} - \eta_j' t_{q_j s_i})\, \beta_{s_i u}\right] = \mathrm{const} \qquad (5.29)$$

in the frequency range of $\sigma_i(i=1,2,\ldots,k)$ and

$$\text{sign}\left[(t_{\psi q_j}-\eta_j')\beta_{q_j u}\right]=\text{const} \qquad (5.30)$$

in the frequency range of $\omega_j(j=1,2,\ldots,m)$.

In the given case the following relation exists between the structural stability and the stability of the space vehicle understood in the classical sense.

In the absence of information about the control system L the signs of the expressions

$$(t_{\psi s_i}-\eta_j' t_{q_j s_i})\beta_{s_i u},\ i=1,2,\ldots,k;\quad (t_{\psi q_j}-\eta_j')\beta_{q_j u},\ i=1,2,\ldots,m$$

determine the requirements on its phase characteristic $\phi(\omega)$, beginning with the necessity for insuring the conditions of stability of the space vehicle (5.27)-(5.28).

For the given characteristics of the control system L, that is, for the known signs of the expressions

$$k_1(\sigma_i)=\frac{1}{\omega_j}A\sin\varphi(\sigma_i);\ k_1(\omega_j)=\frac{1}{\omega_j}A\sin\varphi(\omega_j)$$

for each i, j it is easy to check the satisfiability of the inequalities (5.27)-(5.28). If, for example, the phase conditions of the control system are "uniform," that is, if

$$k_1(\sigma_j)\gtrless 0,\ k_1(\omega_j)\gtrless 0$$

for all $i=1,2,\ldots,k$, $j=1,2,\ldots,m$, then the structural instability of the space vehicle simply means its instability in the classical sense.

Thus, if the parameters of the control system L and the space vehicle are such that for some i (or j) the conditions (5.27) or (5.28) are satisfied, then we talk about the "phase" stabilization of the space vehicle on these frequencies.

The failure to satisfy the phase conditions of stabilization of the space vehicle on any frequencies means the necessity to insure its amplitude stabilization. By this we mean the following.

Denoting the diagonal elements of the matrix of dissipative forces B_k beginning with the third by $\gamma_{s_1},\ldots,\gamma_{s_k}$, $\gamma_{q_1},\ldots,\gamma_{q_m}$, we obtain the following analogs of the expressions (5.24)-(5.25):

$$\operatorname{Re}p_{s_i}=-\frac{\gamma_{s_i}}{2}+\frac{(t_{\psi s_i}-\eta_j' t_{q_j s_i})\beta_{s_i u}k_1(\sigma_i)}{2\sigma_i},\quad i=1,2,\ldots,k; \qquad (5.31)$$

$$\operatorname{Re}p_{q_j}=-\frac{\gamma_{q_j}}{2}+\frac{(t_{\psi q_j}-\eta_j')\beta_{q_j u}k_1(\omega_j)}{2\omega_j},\ j=1,2,\ldots,m, \qquad (5.32)$$

from which we immediately have the conditions of stability of the space vehicle

$$\gamma_{s_i} > (t_{\psi s_i} - \eta_j' t_{q_j s_i})\beta_{s_i u} k_1(\sigma_i) \qquad (5.33)$$

in the frequency band of the fuel oscillations $\{\sigma_i\}$ and

$$\gamma_{q_j} > (t_{\psi q_j} - \eta_j')\beta_{q_j u} k_1(\omega_j) \qquad (5.34)$$

in the frequency band of the elastic vibrations of the spacecraft structure (ω_j).

Since $\gamma_{s_i}>0$, $\gamma_{q_j}>0$ with respect to the meaning of the problem, then the inequalities (5.33)-(5.34) have meaning under the assumption that the system (5.1) (without damping) is unstable (the structural instability effect). On satisfaction of the conditions (5.33)-(5.34), it is said that the amplitude stabilization of the space vehicle will be insured in the investigated frequencies.

In practice the amplitude stabilization of the space vehicle on the frequencies of the fuel oscillations is insured by introducing special intertank devices (various types of baffles). The damping of other vibrations is, as a rule, a less controllable factor.

In conclusion to this section let us present more precisely defined formulas for the roots of the characteristic equation of the system (5.1), close to the partial values of σ_i and ω_j and differing from expressions (5.24), (5.25) by total consideration of the effect of all m forms of elastic vibrations of the space vehicle hull on the stability in the investigated frequency band:

$$\alpha_i = \frac{\left(t_{\psi s_i} - \sum_{j=1}^{m} \eta_j' t_{q_j s_i}\right)\beta_{s_i u} k_1(\sigma_i)}{2\sigma_i}; \quad \sigma_i = \sigma_i^{0'} - \frac{\left(t_{\psi s_i} - \sum_{j=1}^{m} \eta_j' t_{q_j s_i}\right)\beta_{s_i u}}{2} \qquad (5.35)$$

in the range of partial frequencies $\sigma_1^0, \ldots, \sigma_k^0$;

$$\alpha_j = \frac{\left(t_{\psi q_j} - \sum_{k=1}^{m} t_{q_k q_j}\right)\beta_{q_j u} k_1(\omega_j)}{2\omega_j};$$

$$\omega_j = \omega_j^0 - \frac{\left(t_{\psi q_j} - \sum_{k=1}^{m} t_{q_k q_j}\right)\beta_{q_j u} k_0(\omega_j)}{2} \qquad (5.36)$$

in the range of partial frequencies of $\omega_1^0, \ldots, \omega_m^0$.

The values of $k_1(\sigma_i)$; $k_0(\sigma_i)$; $k_1(\omega_j)$; $k_0(\omega_j)$ are calculated by the method of successive approximations beginning with the initial values of $k_1(\sigma_i^0)$, $k_0(\sigma_i^0)$, $k_1(\omega_j^0)$, $k_0(\omega_j^0)$.

Stability of the Space Vehicle with Angular Position Control System (Nonlinear Case)

Let us investigate the characteristic situation where the instability of the space vehicle in the frequency range of the fuel oscillations is caused by the effect of the automatic stabilization system (the case of structural instability of the space vehicle).

The system of equations of the controlled movement of the space vehicle in the active segment will be written in the following form:

$$A\ddot{\vec{x}}+\varepsilon B\dot{\vec{x}}+C\vec{x}=\vec{b}u; \quad (5.37)$$

$$u=L(\psi_2), \quad (5.38)$$

where $\vec{x}=(z, \psi, s_1, \ldots, s_m)$; $\vec{b}=(a_{zu}, a_{\psi u}, 0, \ldots, 0)$,

and the matrices A, C, B are defined in Chapter 3.

Let us reduce the equation (5.37) to the normal form

$$\ddot{\vec{x}}+\varepsilon B_{\text{н}}\dot{\vec{x}}+C_{\text{н}}\vec{x}=\vec{b}_{\text{н}}u; \quad (5.39)$$

$$u=L(x_2), \quad (5.40)$$

calculating the matrices

$$B_{\text{н}}=A^{-1}B;\ C_{\text{н}}=A^{-1}C;\ \vec{b}_{\text{н}}=A^{-1}\vec{b}. \quad (5.41)$$

Considering the structure of the initial matrix B, let us neglect the nondiagonal terms of the matrix obtained B_H. We obtain the following representation of this matrix:

$$E=\text{diag}(0, 0, -\tilde{a}_{11}\varepsilon_1, \ldots, -\tilde{a}_{mm}\varepsilon_m), \quad (5.42)$$

where $\tilde{a}_{ii}$ are the diagonal terms of the matrix A^{-1} corresponding to m additional oscillators in the system (5.37)-(5.38).

Let us propose that the roots of the characteristic equation $|C_H+\lambda^2E|=0$ are real and different: $\lambda_z=-\omega_z^2$; $\lambda_\psi=-\omega_\psi^2$, $\lambda_i=-\omega_i^2$, $i=1,2,\ldots,m$

In this case there is a nondegenerate transformation $\vec{x}=T\vec{y}$ reducing the system (5.39)-(5.40) to the canonical form

$$\ddot{\vec{y}}=\Lambda\vec{y}+\varepsilon B_{\text{н}}\dot{\vec{y}}+\vec{b}_{\text{к}}u; \quad (5.43)$$

$$u=L(\tau' y), \quad (5.44)$$

where

$$\vec{\tau}=(t_{\psi z},\ 1,\ t_{\psi 1},\dots,\ t_{\psi m});\ \vec{b_{\text{н}}}=T^{-1}\vec{b};$$
$$\Lambda=\operatorname{diag}(-\omega_z^2,\ -\omega_\psi^2,\ -\omega_1^2,\dots,\ -\omega_m^2).$$

Just as above, neglecting the nondiagonal terms in the matrix B_H, we obtain

$$B_{\text{н}}=\operatorname{diag}\left(0,\ 0,\ \sum_{j=1}^{m} g_{1j}\varepsilon_j,\dots,\ \sum_{j=1}^{m} g_{mj}\varepsilon_j\right). \qquad (5.45)$$

In the system (5.43)-(5.44) we proceed to the new coordinates

$$\zeta_z=y_z;\ \zeta_\psi=(\vec{\tau}'\vec{y});\ \zeta_j=y_j,\ j=1, 2,\dots, m.$$

Thus, we have:

$$\ddot{\vec{\zeta}}=\Lambda_1\vec{\zeta}+\varepsilon G\dot{\vec{\zeta}}+\vec{\beta}u; \qquad (5.46)$$
$$\dot{u}=L(\zeta_\psi), \qquad (5.47)$$

where

$$\Lambda_1=\begin{pmatrix} -\omega_z^2 & 0 & 0 & \dots & 0 \\ t_{\psi z}(\omega_\psi^2-\omega_z^2) & -\omega_\psi^2 & t_{\psi 1}(\omega_\psi^2-\omega_1^2) & \dots & t_{\psi m}(\omega_\psi^2-\omega_m^2) \\ 0 & 0 & -\omega_1^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\omega_m^2 \end{pmatrix};$$
$$G=B_{\text{н}},\ \beta=(\tau' b_{\text{н}}).$$

Let us consider two types of nonlinearities in the system (5.46)-(5.47), characteristic of the investigated class of objects and primarily the nonlinearity of the servoelements of the space vehicle. Let us set

$$L(\zeta_\psi)=L(p)\zeta_\psi;\ p=i\omega.$$

We shall consider that the following representation is correct

$$L(i\omega)=A(\omega,\ \psi_0)\cos\varphi(\omega,\ \psi_0)+iA(\omega,\ \psi_0)\frac{\sin\varphi(\omega,\ \psi_0)}{\omega}, \qquad (5.48)$$

where $A(\omega, \psi_0)$; $\phi(\omega, \psi_0)$ are the desired characteristics of the nonlinear control system which depend on two parameters: the frequency ω and the amplitude of the input signal ψ_0.

The nonlinearity of the other type is connected with the processes of energy dissipation for fuel oscillations in the tank.

Let us assume that the damping coefficients ε_j depend on the amplitude of the oscillations ζ_{j0} corresponding to the generalized coordinates ζ_j:

$$\varepsilon_j=\nu_j f_j(\zeta_{j0}),\ j=1, 2,\dots, m, \qquad (5.49)$$

where $f(\zeta_{j0})$ are some known functions.

Considering (5.49) we find

$$G=\operatorname{diag}\left(0,\ 0,\ \sum_{j=1}^{m} g_{1j} f_j(\zeta_{j0}), \ldots,\ \sum g_{mj} f_j(\zeta_{0j})\right). \qquad (5.50)$$

Then applying the procedure of the Krylov–Bogolyubov method to the equations (5.46)–(5.47), we obtain the following equations for finding the amplitudes of the limiting cycles:

$$-2a_k+\sum_{j=1}^{m} g_{kj} f_j(\zeta_{j0})=0. \qquad (5.51)$$

Solving the nonlinear equation (5.51) with respect to ζ_{k0}, using the formulas for the inverse conversion $\zeta \to x$ we find the required amplitudes of the limiting cycles for the generalized coordinate z, ψ, $s_1, \ldots, s_m$ on a frequency of ω_k, which solves the stated problem.

Let us note in conclusion that if we use the geometric interpretation of the stabilizability criterion it is possible to note that the possible zone of oscillations will be contained inside the crosshatched region of the plane Z_1, Z_2. Thus, these regions permit interpretation also in the case of the nonlinear statement of the problem of stability of the closed system made up of the space vehicle and the automatic angular stabilization system.

Generalizing everything that has been discussed in this book, let us note the basic conclusion: the investigation of the stabilizability of the space vehicle is a necessary and effective element of solving the problem of insuring stability of the space vehicle in the active segment similarly to how the study of controllability and observability of a system in general control theory is a necessary step in the solution of various problems of optimal control.

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